

THE SIMPLICIAL EHP SEQUENCE IN \mathbb{A}^1 -ALGEBRAIC TOPOLOGY

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ABSTRACT. We give a tool for understanding simplicial desuspension in \mathbb{A}^1 -algebraic topology: we show that $X \rightarrow \Omega(S^1 \wedge X) \rightarrow \Omega(S^1 \wedge X \wedge X)$ is a fiber sequence up to homotopy in 2-localized \mathbb{A}^1 algebraic topology for $X = (S^1)^m \wedge \mathbb{G}_m^{\wedge q}$ with $m > 1$. It follows that there is an EHP sequence spectral sequence

$$\mathbb{Z}_{(2)} \otimes \pi_{n+1+i}^{\mathbb{A}^1}(S^{2n+2m+1} \wedge (\mathbb{G}_m)^{\wedge 2q}) \Rightarrow \mathbb{Z}_{(2)} \otimes \pi_i^{\mathbb{A}^1, s}(S^m \wedge (\mathbb{G}_m)^{\wedge q}).$$

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1. INTRODUCTION

Let Σ denote the suspension functor from pointed simplicial sets (or topological spaces) to itself, defined $\Sigma X := S^1 \wedge X$. For some maps $f : \Sigma Y \rightarrow \Sigma X$, there is a $g : Y \rightarrow X$ such that $f = \Sigma g$. In this case, f is said to desuspend and g is called a desuspension of f . Under certain conditions, the obstruction to desuspending f is a generalized Hopf invariant, as is proven by the existence of the EHP sequence

$$(1) \quad X \rightarrow \Omega \Sigma X \rightarrow \Omega \Sigma^{\wedge 2} X$$

of James [Jam55] [Jam56a] [Jam56b] [Jam57] and Toda [Tod56] [Tod62] which induces a long exact sequence in homotopy groups in a range, see for example [Tod52] or [Whi12, XII Theorem 2.2]. Namely, f desuspends if and only if the generalized Hopf invariant

$$H(f) : Y \rightarrow \Omega \Sigma Y \xrightarrow{\Omega f} \Omega \Sigma X \rightarrow \Omega \Sigma^{\wedge 2} X$$

is null. Because calculations can become easier after applying suspension, it is useful to have such a systematic tool for studying desuspension.

By work of James [Jam56a] [Jam56b], it is known that when X is an odd dimensional sphere, (1) is a fiber sequence, and when X is an even dimensional sphere, (1) is a fiber sequence after localizing at 2. In particular, for any sphere, (1) is a 2-local fiber sequence. Since the suspension of a sphere is again a sphere, the corresponding fiber sequences for all spheres form an exact couple, thereby defining the EHP spectral sequence [Mah82]. The EHP spectral sequence is a tool for calculating unstable homotopy groups of spheres. See for example, the extensive calculations of Toda in [Tod62].

We provide the analogous tools for $\Sigma X = S^1 \wedge X$ in \mathbb{A}^1 -algebraic topology, identifying the obstruction to S^1 -desuspension of a map whose codomain is any sphere with a generalized Hopf invariant, and relating S^1 -stable homotopy groups of spheres to unstable homotopy groups, after 2-localization, by the corresponding EHP special sequence. We leave the p -localized sequence for future work.

Place ourselves in the setting of \mathbb{A}^1 -algebraic topology over a field [MV99] [Mor12]; let \mathbf{Sm}_k denote the category of smooth schemes over a perfect field k , and consider the simplicial model category $\mathbf{sPre}(\mathbf{Sm}_k)$ of

simplicial presheaves on \mathbf{Sm}_k with the \mathbb{A}^1 injective local model structure, which will be recalled in Section 2. This model structure can be localized at a set of primes P (see [Hor06] and Section 3) giving rise to the notation of a P -local fiber sequence up to homotopy. See Definition 8.1. Define the notation

$$S^{n+q\alpha} = (S^1)^{\wedge n} \wedge (\mathbb{G}_m)^{\wedge q}.$$

Let $\Omega(-)$ denote the pointed \mathbb{A}^1 -mapping space $\mathrm{Map}_{\mathbf{sPre}(\mathbf{Sm}_k)_*}(S^1, L_{\mathbb{A}^1} -)$, where $L_{\mathbb{A}^1}$ denotes \mathbb{A}^1 fibrant replacement.

Theorem 1.1. *Let $X = S^{n+q\alpha}$ with $n > 1$. There is a 2-local \mathbb{A}^1 -fiber sequence up to homotopy*

$$X \rightarrow \Omega \Sigma X \rightarrow \Omega \Sigma X^{\wedge 2}.$$

Let $\pi_i^{\mathbb{A}^1}$ denote the i th \mathbb{A}^1 homotopy sheaf, and more generally define $\pi_{i+v\alpha}^{\mathbb{A}^1}(X)$ to be the sheaf associated to the presheaf taking a smooth k -scheme U to the \mathbb{A}^1 -homotopy classes of maps from $S^{i+j\alpha} \wedge U_+$ to X . The stable \mathbb{A}^1 homotopy groups are defined as the colimit $\pi_{i+v\alpha}^{s, \mathbb{A}^1}(X) = \mathrm{colim}_{r \rightarrow \infty} \pi_{i+r+v\alpha}^{\mathbb{A}^1}(\Sigma^r X)$.

Theorem 1.2. *(Simplicial EHP sequence) Choose n, q and v in $\mathbb{Z}_{\geq 0}$ with $n \geq 2$.*

- *There is a spectral sequence $(E_{i,j}^r, d_r : E_{i,j}^r \rightarrow E_{i-1,j-r}^r) \Rightarrow \mathbb{Z}_{(2)} \otimes \pi_{i-n+(v-q)\alpha}^{s, \mathbb{A}^1} = \mathbb{Z}_{(2)} \otimes \pi_{i+v\alpha}^{\mathbb{A}^1, s} S^{n+q\alpha}$ with $E_{i,j}^1 = \mathbb{Z}_{(2)} \otimes \pi_{j+1+i+v\alpha}^{\mathbb{A}^1}(S^{2j+2n+1+2q\alpha})$ if $i \geq 2n-1+j$ and otherwise $E_{i,j}^1 = 0$.*
- *Choose $n' > n$. There is a spectral sequence $(E_{i,j}^r, d_r : E_{i,j}^r \rightarrow E_{i-1,j-r}^r) \Rightarrow \mathbb{Z}_{(2)} \otimes \pi_{i+v\alpha}^{\mathbb{A}^1} S^{n'+q\alpha}$ with $E_{i,j}^1 = \mathbb{Z}_{(2)} \otimes \pi_{j+1+i+v\alpha}^{\mathbb{A}^1}(S^{2j+2n+1+2q\alpha})$ if $i \geq 2n-1+j$ and $j < n'-n$, and $E_{i,j}^1 = 0$ otherwise.*

Theorem 1.2 follows directly from Theorem 1.1. Theorem 1.1 is a summary of a more refined theorem, giving conditions under which (1) is a fiber sequence without 2-localization. To state this theorem, let $\mathrm{GW}(k)$ denote the Grothendieck-Witt group of k , and consider the element of $\mathrm{GW}(k)$ given by $-\langle -1 \rangle = -(1 + \rho\eta)$, where η is the motivic Hopf map and $\rho = [-1]$ in the notation of [Mor12, Definition 3.1]. Let K^{MW} denote Milnor-Witt K-theory defined [Mor12, Definition 3.1]. For a set of primes P , write \mathbb{Z}_P for the ring \mathbb{Z} with formal multiplicative inverses adjoined for all primes not in P .

Theorem 1.3. *Let $X = S^{n+q\alpha}$ with $n > 1$, and let $e = (-1)^{n+q} \langle -1 \rangle^q$. Let P be a set of primes. The sequence*

$$X \rightarrow \Omega \Sigma X \rightarrow \Omega \Sigma X^{\wedge 2}$$

is a P -local \mathbb{A}^1 -fiber sequence up to homotopy if $1 + m(1 + e)$ are units in $\mathrm{GW}(k) \otimes \mathbb{Z}_P$ for all positive integers m .

Corollary 1.4. *In the setting of Theorem 1.3, the sequence*

- *is always a 2-local \mathbb{A}^1 -fiber sequence up to homotopy.*
- *is an \mathbb{A}^1 -fiber sequence up to homotopy when $e = -1$ or when $n + q$ is odd and the field k is not formally real.*

In particular, the sequence is an \mathbb{A}^1 -fiber sequence up to homotopy

- *when n is odd and q is even.*
- *when $n + q$ is odd and $k = \mathbb{C}$, or more generally, when $n + q$ is odd and k is any field such that $2\eta = 0$ in K_*^{MW} .*

Although the statement of Theorem 1.3 is a direct analogue of the corresponding theorem in algebraic topology, the proof given here is not a straightforward generalization of a proof in algebraic topology. The difficulty is that \mathbb{A}^1 -fiber sequences are problematic and \mathbb{A}^1 -homotopy groups are not necessarily finitely generated. Standard tools like the Serre spectral sequence are not currently available.

If a theorem holds for every simplicial set in a functorial manner, it may “globalize” in the the following sense. First, one may be able to obtain in \mathbf{sPre} a naïve analogue by starting with simplicial presheaves

instead of simplicial sets, performing corresponding operations, producing corresponding maps in \mathbf{sPre} . If the theorem in algebraic topology says that some map is always a weak equivalence (respectively weak equivalence through a range), it may be immediate that the corresponding map is a global weak equivalence (respectively global weak equivalence through a range). If the \mathbb{A}^1 -invariant analogues of the operations considered in the theorem are obtained by applying $L_{\mathbb{A}^1}$ to the naïve analogue (defined by applying the operation in simplicial set to the sections over each $U \in \mathbf{Sm}_k$), then the theorem holds in \mathbb{A}^1 -algebraic topology.

This is the case of the Hilton-Milnor splitting shown below:

Theorem 1.5. *There is a natural isomorphism*

$$\Sigma \Omega \Sigma X \rightarrow \Sigma \bigvee_{n=1}^{\infty} X^{\wedge n}$$

in the \mathbb{A}^1 -homotopy category.

This is also the case for the statement that for any simplicial presheaf X , the sequence (1) is a fiber sequence up to homotopy in the range $i \leq 3n - 2$, meaning

$$\begin{aligned} \pi_{3n-2}^{\mathbb{A}^1} X &\rightarrow \pi_{3n-1}^{\mathbb{A}^1} \Sigma X \rightarrow \pi_{3n-1}^{\mathbb{A}^1} \Sigma(X^{\wedge 2}) \rightarrow \dots \\ &\rightarrow \pi_i^{\mathbb{A}^1} X \rightarrow \pi_{i+1}^{\mathbb{A}^1} \Sigma X \rightarrow \pi_{i+1}^{\mathbb{A}^1} \Sigma(X^{\wedge 2}) \rightarrow \pi_{i-1}^{\mathbb{A}^1} X \rightarrow \dots \end{aligned}$$

is exact. This fact is shown in joint work with A. Asok and J. Fasel [AFWB14].

This is not the case for Theorem 1.1 and Theorem 1.3, i.e. these theorems are not proven by globalizing a corresponding result in algebraic topology, where the sequence (1) fails to be exact for $X = S^n \vee S^n$. See Example 6.19.

Here is a sketch of the proof of Theorem 1.1; its purpose is to help the reader understand the proof given in this paper, and also to explain the similarities with, and differences from the situation in classical algebraic topology. Let $J(X)$ denote the free monoid on a pointed object X in simplicial presheaves on \mathbf{Sm}_k , where \mathbf{Sm}_k denotes smooth schemes over a perfect field k .

In algebraic topology, the free monoid on a pointed object is canonically homotopy equivalent to the loops of the suspension. It was understood by Fabien Morel that the same result holds in \mathbb{A}^1 -algebraic topology. Indeed, a result of Morel implies that $L_{\mathbb{A}^1} J(X)$ is simplicially equivalent to $\Omega_{L_{\mathbb{A}^1}} \Sigma X$, for X pointed, fibrant and connected. (The phrase “simplicially equivalent” means weakly equivalent in the injective Nisnevich local model structure. Here, “fibrant” means with respect to this model structure as well.) We show the versions of this result that we need in Section 5. By globalizing a construction from algebraic topology [Whi12, VII §2], there is a sequence

$$X \rightarrow J(X) \rightarrow J(X^{\wedge 2}),$$

where $X \rightarrow J(X)$ is the canonical map induced from the adjunction between Σ and Ω , and $J(X) \rightarrow J(X^{\wedge 2})$ is the James-Hopf map i.e., the above maps exist in \mathbb{A}^1 -algebraic topology and the composite map $X \rightarrow J(X^{\wedge 2})$ is nullhomotopic (simplicially). Thus there is an induced map in the homotopy category from X to the P -localized \mathbb{A}^1 -homotopy fiber of $J(X) \rightarrow J(X^{\wedge 2})$, where P is a set of primes. Use the notation $h : X \rightarrow F$ for this map. Theorems 1.1 and 1.3 say that for X a sphere, h is a P -localized \mathbb{A}^1 -homotopy equivalence for appropriate P , and it is proved as follows.

By Theorem 1.5, there is a map of S^1 -spectra $b : \Sigma^{\infty} J(X) \rightarrow \Sigma^{\infty} X$. Using the tensor structure of spectra over spaces, it follows that there is a map of S^1 -spectra

$$c : \Sigma^{\infty} J(X) \rightarrow \Sigma^{\infty} (X \times J(X^{\wedge 2}))$$

which fits into the commutative diagram

$$\begin{array}{ccc} \Sigma^\infty J(X) & \xrightarrow{\quad} & \Sigma^\infty X \times J(X^{\wedge 2}) \\ & \searrow & \swarrow \\ & \Sigma^\infty J(X^{\wedge 2}) & \end{array}$$

(See Section 6.4, and, for general X , see Section 7.2, in particular, the discussion leading to Equation (22).)

The two spaces $J(X)$ and $X \times J(X^{\wedge 2})$ are the same size in the sense that stably they are both weakly equivalent to $\Sigma^\infty \bigvee_{n=1}^\infty X^{\wedge n}$. To see this, note that $\Sigma^\infty J(X) \cong \Sigma^\infty \bigvee_{n=1}^\infty X^{\wedge n}$ by Theorem 1.5;

$$\Sigma^\infty (X \times J(X^{\wedge 2})) \cong \Sigma^\infty X \vee \Sigma^\infty J(X^{\wedge 2}) \vee \Sigma^\infty (X \wedge J(X^{\wedge 2}))$$

because the product of two spaces is stably equivalent to the wedge of their smash with their wedge, i.e. $\Sigma^\infty (X \times Y) \cong \Sigma^\infty (X \vee Y \vee X \wedge Y)$. By Theorem 1.5, we have stable weak equivalences $J(X^{\wedge 2}) \cong \bigvee_{n=1}^\infty X^{\wedge 2n}$ and $X \wedge J(X^{\wedge 2}) \cong \bigvee_{n=1}^\infty X^{\wedge 2n+1}$. These equivalences, when combined with the previous, show that stably $X \times J(X^{\wedge 2}) \cong \bigvee_{n=1}^\infty X^{\wedge n}$. It is not always the case, however, that the stable map $c : \Sigma^\infty J(X) \rightarrow \Sigma^\infty (X \times J(X^{\wedge 2}))$ constructed above is a weak equivalence, see Example 6.19. In algebraic topology, this map is a weak equivalence for X an odd sphere, and an equivalence after inverting 2 for X an even sphere. We show an analogous fact in \mathbb{A}^1 -algebraic topology, in the following way. By the Hilton-Milnor theorem, the map c can be viewed as a “matrix,” which itself is the product of matrices corresponding to the diagonal of $J(X)$ and a combination of b with the James-Hopf map $J(X) \rightarrow J(X^{\wedge 2})$. Nick Kuhn’s calculations of the stable decomposition of the diagonal of $J(X)$ (see [Kuh01]) and the stable decomposition of the James-Hopf map (see [Kuh87, §6]) in algebraic topology globalize to give the matrix entries of c in terms of sums of permutations of smash powers of X . Morel computes that the swap map $X \wedge X \rightarrow X \wedge X$ is e , and more importantly, any permutation σ on $X^{\wedge m}$ is equivalent to $e^{\text{sign}(\sigma)}$ in the homotopy category (see [Mor12, Lemma 3.43]). Since X is a co-H space, N. Kuhn’s results imply that the matrix entries of c are diagonal, and when combined with Morel’s result, we calculate the n th such entry to be

- $1 + \frac{1}{2} \left(\frac{(2n)!}{2^n n!} - 1 \right) (e + 1)$ for n even.
- $\left(1 + \frac{1}{2} \left(\frac{(2(n-1))!}{2^{n-1} (n-1)!} - 1 \right) (e + 1) \right) \left(\frac{n+1}{2} + \frac{n-1}{2} e \right)$ for n odd.

Note that $(2n)!/(2^n n!) = 1(3)(5) \cdots (2n-1)$ is an odd integer, so that the n th diagonal term of this matrix is of the form $1 + m(e + 1)$, with m an integer, for n even, and a product of two such terms for n odd. Note that $e^2 = 1$ in the homotopy category, because e is the class of the swap. It follows that the product of two terms of the form $(1 + m(e + 1))$ is also of this form because $(e + 1)^2 = 2(1 + e)$. Also note that for any positive integer m , we have that

$$((m+1) + me)((m+1) - me) = 2m + 1,$$

whence $(m+1) + me$ is a unit after localizing at 2. It follows that c is a weak equivalence after 2-localization. More generally, c is a weak equivalence after P -localization whenever all the terms $(m+1) + me$ are units in $\text{GW}(k) \otimes \mathbb{Z}_P$. See Proposition 6.16. This produces the corresponding hypothesis in Theorem 1.3. We can furthermore characterize exactly when $(m+1) + me$ is a unit in $\text{GW}(k)$ for all m : either $e = -1$ or the field is not formally real and $e = -\langle -1 \rangle$. See Corollary 4.8.

We are then in the situation where we have two P -localized \mathbb{A}^1 -fiber sequences

$$(2) \quad F \rightarrow J(X) \rightarrow J(X^{\wedge 2})$$

$$(3) \quad X \rightarrow X \times J(X^{\wedge 2}) \rightarrow J(X^{\wedge 2}),$$

and a stable equivalence between the total spaces which respects the maps to the base. We would like to “cancel off” the base $J(X^{\wedge 2})$ to conclude that there is an equivalence between the fibers. This is indeed what we do, however, there are two major obstacles to overcome with this approach.

The first is that the standard tool to measure the size of a fiber of a fibration in terms of the base and total space is the Serre spectral sequence, and at present there is no Serre spectral sequence for \mathbb{A}^1 -fiber sequences. The desired such sequence would use a homology theory like $H_*^{\mathbb{A}^1}$ (see [Mor12, Definition 6.29]) because of the need for analogues of the Hurewicz theorem as in [Mor12, Chapter 6.3] to conclude a weak equivalence between the fibers. We use S^1 -stable \mathbb{A}^1 homotopy groups on the obvious analogue of the Serre spectral sequence defined by lifting the skeletal filtration on the base to express the total space as a filtered limit of cofibrations, and then making an exact couple by applying π_i^{s,P,\mathbb{A}^1} . This gives a spectral sequence even for a global fibration, but it is not clear that it can be controlled. We provide some of the desired control in Section 7.2. Assume for simplicity that the base is reduced in the sense that its 0-skeleton is a single point, as is the case for $J(X^{\wedge 2})$. The E^1 -page can be then identified with π_i^{s,P,\mathbb{A}^1} applied to a wedge indexed by the non-degenerate simplices of the base of the fibration. This wedge construction takes P -local \mathbb{A}^1 -weak equivalences of the fiber (respectively P -local \mathbb{A}^1 -weak equivalences in a range) to P - \mathbb{A}^1 weak equivalences (respectively in a range). See Lemmas 7.12, 7.15, and 7.16. We then show that this identification of the E^1 -page is natural with respect to maps, and even natural with respect to the stable map c discussed above. See Lemma 7.13. This identification of the E^1 -page does not behave well with respect to weak equivalences of the base, as it involves the specific simplices of the base. It is sufficient here because the map on the base is the identity. We do not understand the E^2 page.

We then have a map of spectral sequences from the spectral sequence associated to (2) to the spectral sequence associated to (3). We wish to use this map of spectral sequences to show that the stable weak equivalences of the base and total space imply a stable \mathbb{A}^1 equivalence of the fibers, after appropriately localizing.

Then comes the second difficulty. There are infinitely many non-vanishing stable homotopy groups of the fibers in question, and these groups themselves are not necessarily finitely generated abelian groups. We need to show that there is an isomorphism of these E^1 -pages, but to do this, we need to allow for the possibility that all terms of both spectral sequences are non-zero non-finitely generated groups. We give an inductive argument to do this in Proposition 7.19, and immediately following the proposition there is a verbal description of what happened.

The strategy of this proof of the motivic EHP sequence is modeled on the proof of the EHP sequence given in Michael Hopkins’s stable homotopy course at Harvard University in the fall of 2012. Hopkins credits this proof to James [Jam55] [Jam56b] together with some ideas of Ganea [Gan68]. In this argument, the original Serre spectral sequence is used; there is no need to work in spectra, as calculations in (co)homology suffice. Since the (co)homology of spheres in algebraic topology is concentrated in two degrees, there is no analogue of Proposition 7.19.

It is also possible to compute the first differential in the EHP sequence of Theorem 1.2, and this computation will be made available in a joint paper with Asok, Fasel and the present authors [AFWB14].

Computations of unstable motivic homotopy groups of spheres can be applied to classical problems in the theory of projective modules, for example to the problem of determining when algebraic vector bundles decompose as a direct sum of algebraic vector bundles of smaller rank. See [Mor12, Chapter 8], [AF14b], and [AF14a].

In a different direction, it can be shown that there is an \mathbb{A}^1 weak equivalence $\Sigma(\mathbb{P}^1 - \{0, 1, \infty\}) \cong \Sigma(\mathbb{G}_m \vee \mathbb{G}_m)$ between the S^1 suspensions of $\mathbb{P}^1 - \{0, 1, \infty\}$ and $\mathbb{G}_m \vee \mathbb{G}_m$. Restrict attention to the case where k is a number field, and change topologies to the étale topology. By comparing the actions of the absolute Galois group

on geometric étale fundamental groups, it can be shown that any chosen weak equivalence between the suspensions does not desuspend. Because the action of the absolute Galois group on $\pi_1^{\text{ét}}(\mathbb{P}_{\mathbb{Q}}^1 - \{0, 1, \infty\})$ is both tied to interesting mathematics [Iha91] and obstructs desuspension, it is potentially also of interest to have systematic tools like those provided by the EHP sequence to study the obstructions to desuspension.

1.1. Organization. The organization of this paper is as follows: Theorem 1.2 is proven in Section 8 as Theorems 8.5 and 8.6. Theorem 1.1 is proven in Section 8 as Theorem 8.3. The core of these arguments is the cancelation property of Section 7.3. The substitute for the Serre spectral sequence is developed in Section 7. In Section 6, the motivic James-Hopf map and the diagonal of the James construction are computed stably as matrices with entries in $\text{GW}(k)$. Section 5 proves the Hilton-Milnor splitting. Section 4 gives results on the Grothendieck–Witt group that are needed to understand when the matrices computed in Section 6 are invertible. Section 3 provides needed results on localizations of $\mathbf{sPre}(\mathbf{Sm}_k)$ and $\mathbf{Spt}(\mathbf{Sm}_k)$, and Section 2 introduces the needed notation and background on \mathbb{A}^1 -homotopy theory.

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2. OVERVIEW OF \mathbb{A}^1 HOMOTOPY THEORY

Let k be a field such that the results of [Mor12] hold over $\text{Spec } k$. It suffices that k be perfect, but we hope that in future this requirement may be weakened.

Let \mathbf{Sm}_k denote a small category equivalent to the category of smooth, finite type k -schemes. The category $\mathbf{sPre}(\mathbf{Sm}_k)$ is the category of simplicial presheaves on \mathbf{Sm}_k , and $\mathbf{sPre}(\mathbf{Sm}_k)_*$ the category of pointed simplicial presheaves. The category \mathbf{Sm}_k is considered embedded in $\mathbf{sPre}(\mathbf{Sm}_k)$ via the Yoneda embedding. The terminal object of \mathbf{Sm}_k and $\mathbf{sPre}(\mathbf{Sm}_k)$ is therefore $\text{Spec } k$, which is also denoted by k and $*$ depending on the context.

The notation $\text{Map}(X, Y)$ denotes the internal mapping object where it appears, generally in $\mathbf{sPre}(\mathbf{Sm}_k)$. The notation $\text{SMap}(X, Y)$ denotes a simplicial mapping object, which is a simplicial set.

2.1. Model Structures. This paper makes use of two families of model structure on the category $\mathbf{sPre}(\mathbf{Sm}_k)$ and its descendants. In the first place, the local injective model structure of [Jar87]—introduced there as the ‘global’ model structure—and the local flasque model structure of [Isa05]. These are Quillen equivalent. Each gives rise to descendent model structures by \mathbb{A}^1 – or P – localization or by stabilization. The flasque model structures are employed to prove technical results regarding spectra; when ‘flasque’ is not specified, it is to be understood that the injective structures are meant.

The weak equivalences in the injective local and the flasque local model structures are the local weak equivalences—those maps that induce isomorphisms on homotopy sheaves, properly defined: [Jar87]. In the seminal work [MV99], these maps are called ‘simplicial weak equivalences’ in order to emphasise their non-algebraic character.

Both the injective and the flasque local model structures are left Bousfield localizations of global model structures on $\mathbf{sPre}(\mathbf{Sm}_k)$; a global model structure being one where the weak equivalences are those maps $\phi : X \rightarrow Y$ inducing weak equivalences $\phi(U) : X(U) \rightarrow Y(U)$ for all objects U of \mathbf{Sm}_k . Both the global

injective and the global flasque model structures are left proper, simplicial, cellular (see [Hor06]) and combinatorial so that left Bousfield localizations of either at any set of morphisms exist and are again left proper, simplicial and cellular. In the injective model structures all objects are cofibrant, and therefore these model structures are tractable in the sense of [Bar10].

The injective global model structure is symmetric monoidal, the structure being given by \times and the usual internal mapping object, this being the evident extension of the symmetric monoidal structure on \mathbf{sSet} . By [Bar10, 4.46], any left Bousfield localization, \mathfrak{a} , of the injective local model structure will inherit the structure of a module over the injective local model structure. In particular, any object X of $\mathbf{sPre}(\mathbf{Sm}_k)_*$ gives rise to a left Quillen functor $X \times \cdot : \mathbf{sPre}(\mathbf{Sm}_k) \rightarrow \mathbf{sPre}(\mathbf{Sm}_k)$ where $\mathbf{sPre}(\mathbf{Sm}_k)$ is endowed with the structure \mathfrak{a} . Since $X \times \cdot$ preserves trivial cofibrations and all objects are cofibrant, by Ken Brown's lemma it preserves weak equivalences.

Lemma 2.1. *Suppose \mathfrak{a} is a left Bousfield localization of the injective global model structure on $\mathbf{sPre}(\mathbf{Sm}_k)$, and suppose X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)$. The functor $X \times \cdot$ preserves \mathfrak{a} weak equivalences.*

These model structures all have pointed analogs, and a standard argument allows us to deduce:

Corollary 2.2. *Suppose \mathfrak{a}_* is a model structure on $\mathbf{sPre}(\mathbf{Sm}_k)_*$ that is associated to a left Bousfield localization of the injective global model structure on $\mathbf{sPre}(\mathbf{Sm}_k)$, and suppose X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$. The functor $X \wedge \cdot$ preserves \mathfrak{a} weak equivalences.*

Proof. Let $f : Z \rightarrow Y$ be a \mathfrak{a} weak equivalence. Because \mathfrak{a}_* is a simplicial model category in which all objects are cofibrant and monomorphisms are cofibrations, it follows from [Rie14, Corollary 14.3.2] that $\text{id}_X \vee f : X \vee Z \rightarrow X \vee Y$ is a \mathfrak{a} weak equivalence.

By Lemma 2.1, $X \times f : X \times Z \rightarrow X \times Y$ is a \mathfrak{a} weak equivalence. Note that $\text{id}_X \vee f$, id_* , $X \times f$, and $X \wedge f$ determine a map of push-out squares as in the commutative diagram

$$\begin{array}{ccccc}
 & & * & \xrightarrow{\quad} & X \wedge Z \\
 & \nearrow & \downarrow & & \downarrow \\
 X \vee Z & \xrightarrow{\quad} & X \times Z & \xrightarrow{\quad} & X \wedge Z \\
 \downarrow & & \downarrow & & \downarrow \\
 & \nearrow & * & \xrightarrow{\quad} & X \wedge Y \\
 X \vee Y & \xrightarrow{\quad} & X \times Y & \xrightarrow{\quad} & X \wedge Y
 \end{array}$$

Furthermore, $X \vee Z \rightarrow X \times Z$ and $X \vee Y \rightarrow X \times Y$ are cofibrations because they are monomorphisms. It now follows from [Rie14, Corollary 14.3.2] that $X \wedge f : X \wedge Z \rightarrow X \wedge Y$ is a \mathfrak{a} weak equivalence as claimed. \square

2.2. Homotopy Sheaves. If X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)$ or $\mathbf{sPre}(\mathbf{Sm}_k)_*$, we write $L_{\text{Nis}}X$ for a functorial fibrant replacement in the local injective model structure, and $L_{\text{Nis}}^{\text{fl}}X$ in the local flasque model structure. We write $L_{\mathbb{A}^1}$ or $L_{\mathbb{A}^1}^{\text{fl}}$ for a functorial fibrant replacement in the appropriate \mathbb{A}^1 model structures.

Since the purpose of this paper is to establish some identities regarding \mathbb{A}^1 homotopy sheaves, it behoves us to define what a homotopy sheaf means in the sequel.

The following definitions date at least to [Jar87].

Definition 2.3. If X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)$, then we define $\pi_0^{\text{pre}}(X)$ as the presheaf

$$U \mapsto \pi_0(|X(U)|)$$

where \mathcal{U} is an object of \mathbf{Sm}_k , and where $|X(\mathcal{U})|$ indicates a geometric realization of $X(\mathcal{U})$. We define $\pi_0(X)$ as the associated Nisnevich sheaf to $\pi_0^{\text{pre}}(X)$.

Definition 2.4. If X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, with basepoint $x_0 \rightarrow X$, then we define $\pi_i^{\text{pre}}(X, x_0)$ for $i \geq 1$ as the presheaf

$$\mathcal{U} \mapsto \pi_i(|X(\mathcal{U})|, x_0)$$

where \mathcal{U} is an object of \mathbf{Sm}_k , and where x_0 , in an abuse of notation, indicates the basepoint of $|X(\mathcal{U})|$ induced by $x_0 \rightarrow X$. We define $\pi_i(X, x_0)$ as the associated Nisnevich sheaf. The basepoint x_0 will generally be understood and omitted.

The reader is reminded that $X(\mathcal{U})$ may have connected components that do not appear in the global sections, $X(*)$. In this case, the groups $\pi_i(X, x_0)$ as defined above are insufficient to describe the homotopy of X .

It is the case that the functor $\pi_i(\cdot)$ takes simplicial weak equivalences to isomorphisms, and $\pi_0^{\mathbb{A}^1}$ takes \mathbb{A}^1 weak equivalences to isomorphisms.

If K is a simplicial set, then we write K_i for the set of i -simplices in K .

Proposition 2.5. If X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)$ such that $X(\mathcal{U})$ is a fibrant simplicial set for all objects \mathcal{U} of \mathbf{Sm}_k , then $\pi_0(X)$ is the sheaf associated to the presheaf coequalizer:

$$\mathcal{U} \mapsto \text{coeq} \left(X(\mathcal{U})_1 \begin{array}{c} \xrightarrow{d_1} \\ \xrightarrow{d_0} \end{array} X(\mathcal{U})_0 \right).$$

If X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, we reserve the notation $\Omega^i X$ for the derived loop space $\text{Map}_*(S^n, L_{\text{Nis}} X)$. In particular, $\Omega^0 X \cong L_{\text{Nis}} X$.

When there is a model structure present on $\mathbf{sPre}(\mathbf{Sm}_k)_*$, the notation $[X, Y]$ denotes the set of maps in the homotopy category from X to Y , where X, Y are objects of $\mathbf{sPre}(\mathbf{Sm}_k)_*$.

We rely on the following result throughout.

Proposition 2.6. Equip $\mathbf{sPre}(\mathbf{Sm}_k)_*$ with the Nisnevich local model structure. If X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, and if $i \geq 0$, then $\pi_i(X)$ is the sheaf associated to the presheaf

$$\mathcal{U} \mapsto [\Sigma^i(\mathcal{U}_+), X].$$

Proof. We may assume X has been replaced by a weakly equivalent fibrant object. Then the stated presheaf is isomorphic to

$$\pi_0^{\text{pre}}(\text{Map}_*(S^i, X)) \cong \pi_0^{\text{pre}}(\Omega^i X) \cong \pi_i^{\text{pre}}(L_{\text{Nis}} X),$$

and the result follows. \square

Corollary 2.7. If X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, and if $i \geq 0$, then $\pi_i(X) \cong \pi_0(\Omega^i X)$.

Since taking global sections, $X \mapsto X(k)$, is taking a stalk, we have the following corollary:

Corollary 2.8. If X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, and if $i \geq 0$, then

$$\pi_i(X)(k) \cong [S^i, X].$$

If $i, j \geq 0$ we define

$$S^{i+j\alpha} = S^i \wedge \mathbb{G}_m^{\wedge j},$$

where \mathbb{G}_m is pointed at the rational point 1. If $j \geq 0$ and X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)_+$, we define $\pi_{i+j\alpha}^{\mathbb{A}^1}(X)$ as $\pi_i(\text{Map}_*(\mathbb{G}_m^{\wedge j}, L_{\mathbb{A}^1} X))$; it is isomorphic to the sheaf associated to the presheaf $\mathcal{U} \mapsto [S^{i+j\alpha} \wedge \mathcal{U}_+, X]_{\mathbb{A}^1}$. Taking

global sections, we have

$$\pi_{i+j\alpha}^{\mathbb{A}^1}(X)(k) \cong [S^{i+j\alpha}, X]_{\mathbb{A}^1}.$$

2.3. Compact Objects and Flasque Model Structures. We say that an object X of $\mathbf{sPre}(\mathbf{Sm}_k)_*$ is *compact* if

$$\operatorname{colim}_i \operatorname{Map}_*(X, F_i) \cong \operatorname{Map}_*(X, \operatorname{colim}_i F_i)$$

whenever F_i is a filtered system in $\mathbf{sPre}(\mathbf{Sm}_k)$, and similarly for $\mathbf{sPre}(\mathbf{Sm}_k)_*$. An argument similar to that of [DI05, Lemma 9.13] shows that pointed smooth schemes are compact, and it is easy to see that finite constant simplicial presheaves are compact. If A, B are compact objects, then $A \wedge B$ is compact, and all finite colimits of compact objects are again compact.

Proposition 2.9. *If X_i is a filtered diagram in $\mathbf{sPre}(\mathbf{Sm}_k)$, then the natural map of sheaves*

$$\operatorname{colim}_i \pi_0(X_i) \rightarrow \pi_0(\operatorname{colim}_i X_i)$$

is an isomorphism.

Proof. Since the diagram is filtered and all objects are injective cofibrant, it follows that $\operatorname{colim}_i X_i \simeq \operatorname{hocolim}_i X_i$. In particular, both $\pi_0 \operatorname{colim}_i X_i$, and $\operatorname{colim}_i \pi_0 X_i$ are unchanged if we replace the diagram X_i by a termwise weakly equivalent one. We can therefore assume that $X_i(U)$ is a fibrant simplicial set for all objects U of \mathbf{Sm}_k .

Since, according to Proposition 2.5, $\pi_0(Y)$ is the sheaf associated to a coequalizer of presheaves, provided Y takes values in fibrant simplicial sets, the result follows by commuting colimits. \square

The injective local model structure on $\mathbf{sPre}(\mathbf{Sm}_k)_*$ suffers from a technical drawback when one wishes to calculate with filtered colimits, which is that filtered colimits of fibrant objects are not necessarily fibrant themselves. This is the problem that motivates the construction of the flasque model structures of [Isa05], and one can see the presence of flasque or flasque-like conditions appearing often throughout the literature when calculations with filtered colimits are being carried out, see [Jar00], [DI05], [Mor05].

We therefore consider two flasque model structures on $\mathbf{sPre}(\mathbf{Sm}_k)$: the local flasque structure in which the weak equivalences are the simplicial weak equivalences, and the \mathbb{A}^1 flasque structure in which the weak equivalences are the \mathbb{A}^1 weak equivalences. These model structures apply also to $\mathbf{sPre}(\mathbf{Sm}_k)_*$. These model structures are simplicial, proper and cellular, and the \mathbb{A}^1 structures are left Bousfield localizations of the local model structure. There is a square of Quillen adjunctions

$$(4) \quad \begin{array}{ccc} \text{Injective Local} & \xrightarrow{\simeq} & \text{Flasque Local} \\ \downarrow & & \downarrow \\ \text{Injective } \mathbb{A}^1 & \xrightarrow{\simeq} & \text{Flasque } \mathbb{A}^1 \end{array}$$

where the arrows indicate the left adjoints, and each arrow is the identity functor on $\mathbf{sPre}(\mathbf{Sm}_k)$. The horizontal arrows represent Quillen equivalences. A similar diagram obtains for $\mathbf{sPre}(\mathbf{Sm}_k)_*$.

Not all objects are cofibrant in $\mathbf{sPre}(\mathbf{Sm}_k)$ or $\mathbf{sPre}(\mathbf{Sm}_k)_*$ in the flasque model structures, in contrast with the case of the injective structures. Since the \mathbb{A}^1 flasque structures are left Bousfield localizations of the local flasque structures, the cofibrant objects in one model structure agree with the cofibrant objects in the other. The results of [Isa05], specifically Lemmas 3.13, 6.2, show that all pointed simplicial sets and all quotients X/Y of monomorphisms $Y \rightarrow X$ in \mathbf{Sm}_k are flasque cofibrant in $\mathbf{sPre}(\mathbf{Sm}_k)_*$. This includes all smooth schemes pointed at a rational point. Lemma 3.14 of [Isa05] shows that finite smash products of flasque cofibrant objects are again flasque cofibrant in $\mathbf{sPre}(\mathbf{Sm}_k)_*$.

Proposition 2.10. *If F_i is a filtered diagram of fibrant objects in the injective local model structure (resp. injective \mathbb{A}^1 model structure) on $\mathbf{sPre}(\mathbf{Sm}_k)_*$, and if X is a compact and flasque cofibrant object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, then there is a zigzag of local (resp. \mathbb{A}^1) weak equivalences:*

$$(5) \quad \operatorname{colim}_i \operatorname{Map}_*(X, RF_i) \rightarrow \operatorname{Map}_*(X, R \operatorname{colim}_i RF_i) \leftarrow \operatorname{Map}_*(X, R \operatorname{colim}_i F_i),$$

where R denotes an injective local (resp. injective \mathbb{A}^1) functorial fibrant replacement.

Proof. The colimit of a filtered system agrees, up to homotopy, with the homotopy colimit calculated in an injective model structure, where all objects are cofibrant. In particular, the local (resp. \mathbb{A}^1) homotopy type of a filtered colimit is invariant under termwise replacement by locally (resp. \mathbb{A}^1) equivalent objects.

Filtered colimits of flasque fibrant objects are again flasque fibrant, see [Isa05].

The objects RF_i are flasque fibrant, so the colimit $\operatorname{colim}_i RF_i$ is flasque fibrant, as is $R \operatorname{colim}_i RF_i$. There is a global weak equivalence $\operatorname{colim}_i RF_i \simeq R \operatorname{colim}_i RF_i$. Since R preserves weak equivalences, we also have $R \operatorname{colim}_i RF_i \simeq R \operatorname{colim}_i F_i$. Since the object X is flasque cofibrant, the functor $\operatorname{Map}_*(X, \cdot)$ preserves trivial flasque fibrations, and by Ken Brown's lemma, weak equivalences between flasque fibrant objects. The map $\operatorname{Map}_*(X, \operatorname{colim}_i RF_i) \rightarrow \operatorname{Map}_*(X, R \operatorname{colim}_i F_i)$ is therefore a weak equivalence. The result now follows from the compactness of X . \square

Corollary 2.11. *If F_r is a filtered system of objects of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, and if $i, j \geq 0$ are integers, then there are natural isomorphisms of sheaves*

$$\pi_i(\operatorname{colim}_r F_r) \cong \operatorname{colim}_r \pi_i(F_r)$$

and

$$\pi_{i+j\alpha}^{\mathbb{A}^1}(\operatorname{colim}_r F_r) \cong \operatorname{colim}_r \pi_{i+j\alpha}^{\mathbb{A}^1}(F_r).$$

Proof. Combine Corollary 2.7 and Propositions 2.9, 2.10, noting that the objects $S^{i+j\alpha}$ are compact and flasque cofibrant. \square

We warn the reader that $\pi_{i+j\alpha}^{\mathbb{A}^1}(\Omega^+ X)$ differs from $\pi_{i+r+j\alpha}^{\mathbb{A}^1}(X)$ in general, [Mor12, Theorem 6.46].

2.4. Spectra. We take [Hov01] as our main reference for the theory of spectra in model structures such as those we consider here. We shall require only naïve spectra, rather than symmetric spectra. For us a spectrum, E , shall be an S^1 -spectrum, consisting of a sequence $\{E_i\}_{i=0}^\infty$ of objects of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, equipped with bonding maps $\sigma : \Sigma E_i \rightarrow E_{i+1}$. The maps of spectra $E \rightarrow E'$ being defined as levelwise maps $E_i \rightarrow E'_i$ which furthermore commute with the bonding maps, we have a category of presheaves of spectra, which we denote by $\mathbf{Spt}(\mathbf{Sm}_k)$.

Just as we have two notions of weak equivalence on $\mathbf{sPre}(\mathbf{Sm}_k)_*$, the local and the \mathbb{A}^1 , we shall have two kinds of weak equivalence between objects of $\mathbf{Spt}(\mathbf{Sm}_k)$, the stable and the \mathbb{A}^1 .

There is a set, I in the notation of [Isa05], of generating cofibrations for which the domains and codomains all possess the property that we call “compact”, which [Isa05] calls “ ω -small” and which is stronger than the property that [Hov01] calls “finitely presented”. Moreover, both model structures are localizations of an objectwise flasque model structure having a set, J in the notation of [Isa05], which again consists of maps having finitely-presented domains and codomains. By the arguments of [Hov01, Section 4], these model structures are *almost finitely generated*.

The theory of [Hov01, Section 3] establishes a stable model structure on $\mathbf{Spt}(\mathbf{Sm}_k)$ based on any cellular, left proper model structure, \mathfrak{a} , on $\mathbf{sPre}(\mathbf{Sm}_k)_*$. In particular, this applies when \mathfrak{a} is a left Bousfield localization of the global injective or global flasque model structure, and therefore when it is one of the four structures

of (4). The results of [Hov01, Section 5] ensure that we have Quillen adjunctions and equivalences between these model structures:

$$(6) \quad \begin{array}{ccc} \text{Stable Injective Local} & \xrightarrow{\simeq} & \text{Stable Flasque Local} \\ \downarrow & & \downarrow \\ \text{Stable Injective } \mathbb{A}^1 & \xrightarrow{\simeq} & \text{Stable Flasque } \mathbb{A}^1. \end{array}$$

Since the functors of (4) are the identity functors, the same is true of the functors of (6); only the model structure varies.

We write *stable weak equivalence* for the weak equivalences of the stable injective local and stable flasque local model structures, and *stable \mathbb{A}^1 equivalence* for the weak equivalences of the stable injective \mathbb{A}^1 and the stable flasque \mathbb{A}^1 model structures. In keeping with our convention, we write $L_{\mathbb{A}^1} E$ to denote a fibrant replacement of E in the stable \mathbb{A}^1 model structures.

Since the underlying unstable model structures are proper, we may apply fibrant-replacement functors levelwise to objects in $\mathbf{Spt}(\mathbf{Sm}_k)$ to obtain maps of spectra: $E \rightarrow RE$ given by $E_i \rightarrow RE_i$, the fibrant replacement in any one of the four unstable model structures under consideration. There is also a spectrum-level infinite loop space functor, Θ^∞ that takes a spectrum E to the spectrum having i -th space

$$(\Theta^\infty E)_i = \operatorname{colim}_{k \rightarrow \infty} \operatorname{Map}_*(S^k, E_{i+k}).$$

Proposition 2.12. *A map $f : E \rightarrow E'$ of $\mathbf{Spt}(\mathbf{Sm}_k)$ is a stable weak equivalence (resp. a stable \mathbb{A}^1 equivalence) if and only if*

$$\Theta^\infty(Wf)_i : (\Theta^\infty WE)_i \rightarrow (\Theta^\infty WE')_i$$

is a weak equivalence for all i , where W represents the flasque local fibrant replacement functor (resp. flasque \mathbb{A}^1 fibrant replacement functor).

Proof. This is a special case of [Hov01, Theorem 4.12]. The ancillary hypotheses given there, that sequential colimits in commute with finite products and that $\operatorname{Map}_*(S^1, \cdot)$ commutes with sequential limits, are satisfied in $\mathbf{sPre}(\mathbf{Sm}_k)_*$. \square

One can verify that a spectrum E is weakly equivalent to the spectrum one obtains from E by replacing each space E_i by the connected component of the basepoint in E_i . We may therefore assume that E_i is connected, meaning we do not have to worry about the problem of non-globally-defined basepoints.

For any integer i , there is an adjunction of categories

$$(7) \quad \Sigma^{\infty-i} : \mathbf{sPre}(\mathbf{Sm}_k)_* \rightleftarrows \mathbf{Spt}(\mathbf{Sm}_k) : \operatorname{Ev}_i$$

The spectrum $\Sigma^{\infty-i} X$ is the spectrum the j -th space of which is $\Sigma^{j-i} X$ if $j \geq i$, and $*$ otherwise, and where the bonding maps are the evident ones. The right adjoint Ev_i takes E to E_i .

Proposition 2.13. *Suppose α is a left Bousfield localization of either the global injective or the global flasque model structure on $\mathbf{sPre}(\mathbf{Sm}_k)$. Then the adjoint functors of (7) form a Quillen pair between the pointed model structure on $\mathbf{sPre}(\mathbf{Sm}_k)_*$ and the stable model structure on $\mathbf{Spt}(\mathbf{Sm}_k)$ induced by α .*

Proof. This follows from Definition 1.2, Proposition 1.15 and Definition 3.3 of [Hov01]. \square

The left-derived functor of Ev_0 in the flasque model structures are the functors

$$\Omega^\infty : \{E_n\}_n \mapsto \operatorname{colim}_k \Omega^k WE_{n+k}$$

where W is either $L_{\text{Nis}}^{\text{fl}}$ or $L_{\mathbb{A}^1}^{\text{fl}}$, as appropriate. The smash product on $\mathbf{sPre}(\mathbf{Sm}_k)_*$ extends to an action of $\mathbf{sPre}(\mathbf{Sm}_k)_*$ on $\mathbf{Spt}(\mathbf{Sm}_k)$, and the functors $\Sigma^{\infty-i}$ preserve this structure, and in particular are simplicial functors, [Hov01, Section 6].

For an object E of $\mathbf{Spt}(\mathbf{Sm}_k)$, we define the stable homotopy sheaves $\pi_i^s(E)$ as the colimit

$$\pi_i^s(E) = \operatorname{colim}_{r \rightarrow \infty} \pi_{i+r}(E_r).$$

Since $\pi_{i+r}(E_r)$ is the sheaf associated to the presheaf $U \mapsto \pi_{i+r}(E_r(U))$, and sheafification commutes with colimits, it follows that $\pi_i^s(E)$ may also be described as the sheaf associated to the presheaf

$$U \mapsto \pi_i^s(E(U))$$

where the stable homotopy group is the ordinary stable homotopy group of simplicial spectra. This definition of the sheaf π_i^s is used in [Mor05].

We similarly define the stable \mathbb{A}^1 homotopy sheaves $\pi_{i+j\alpha}^{s, \mathbb{A}^1}(E)$ as the colimit

$$\pi_{i+j\alpha}^{s, \mathbb{A}^1}(E) = \operatorname{colim}_{r \rightarrow \infty} \pi_{i+r+j\alpha}^{\mathbb{A}^1}(E_r).$$

Proposition 2.14. *Let $f : E \rightarrow E'$ be a map in $\mathbf{Spt}(\mathbf{Sm}_k)$. Then*

- (1) *f is a stable weak equivalence if and only if $\pi_i^s(f)$ is an isomorphism for all i ;*
- (2) *f is an \mathbb{A}^1 stable weak equivalence if and only if $\pi_{i+j\alpha}^{s, \mathbb{A}^1}(f)$ is an isomorphism for all i .*

Proof. The map $f : E \rightarrow E'$ is a stable weak equivalence if and only if the maps $(\Theta^\infty L_{\text{Nis}}^{\text{fl}} E)_i \rightarrow (\Theta^\infty L_{\text{Nis}}^{\text{fl}} E')_i$ are simplicial weak equivalences for all i . The space $(\Theta^\infty L_{\text{Nis}}^{\text{fl}} E)_j$ is

$$\operatorname{colim}_{r \rightarrow \infty} \Omega^r(L_{\text{Nis}}^{\text{fl}} E_{j+r})$$

and its $i+j$ -th homotopy sheaf is, by Corollary 2.11,

$$\pi_{i+j}(\operatorname{colim}_{r \rightarrow \infty} \Omega^r(L_{\text{Nis}}^{\text{fl}} E_{j+r})) \cong \operatorname{colim}_{j+r \rightarrow \infty} \pi_{i+j+r}(E_{j+r}) \cong \pi_i^s(E).$$

The result for π_i^s follows.

For \mathbb{A}^1 equivalence, the same argument applies *mutatis mutandis*. Writing $L_{\mathbb{A}^1}^{\text{fl}}$ for the flasque \mathbb{A}^1 fibrant replacement functor, we see that the $i+j$ -th homotopy sheaf of the j -th level of the \mathbb{A}^1 stable fibrant replacement $\Theta^\infty(L_{\mathbb{A}^1}^{\text{fl}} E)$ is $\pi_{i+j}(\operatorname{colim}_{r \rightarrow \infty} \Omega^r(L_{\mathbb{A}^1}^{\text{fl}} E_{j+r}))$ which simplifies to $\pi_i^s(L_{\mathbb{A}^1}^{\text{fl}} E) = \pi_i^{s, \mathbb{A}^1}(E)$ \square

This proposition says that the definition of stable weak equivalence used in this paper agrees with that of [Mor05].

Proposition 2.15. *For any object E of $\mathbf{Spt}(\mathbf{Sm}_k)$ and any nonnegative integers i, i' and j ,*

- (1) *The sheaf associated to the presheaf*

$$U \mapsto [\Sigma^{\infty-i'}(S^i \wedge U_+), E]$$

is $\pi_{i-i'}^s(E)$

- (2) *The sheaf associated to the presheaf*

$$U \mapsto [\Sigma^{\infty-i'}(S^{i+j\alpha} \wedge U_+), E]_{\mathbb{A}^1}$$

is $\pi_{i-i'+j\alpha}^{s, \mathbb{A}^1}(E)$.

Proof. We prove the first statement.

The given presheaf, by adjunction, is

$$\mathcal{U} \mapsto [S^i \wedge \mathcal{U}_+, \text{Ev}_i, E],$$

where the functor Ev_i is a derived functor in the flasque stable model structure. By reference to [Hov01], we write this presheaf more explicitly as

$$\mathcal{U} \mapsto [S^i \wedge \mathcal{U}_+, \text{colim}_{r \rightarrow \infty} \Omega^r L_{\text{Nis}}^{\text{fl}} E_{i'+r}]$$

which is associated to the sheaf

$$\pi_i(\text{colim}_{r \rightarrow \infty} \Omega^r L_{\text{Nis}}^{\text{fl}} E_{i'+r}) \cong \text{colim}_{r \rightarrow \infty} \pi_{i-i'+(i'+r)}(E_{i'+r}) \cong \pi_{i-i}^s(E),$$

as asserted.

The proof of the second statement is similar, with the proviso that $L_{\text{Nis}}^{\text{fl}}$ is replaced by $L_{\mathbb{A}^1}^{\text{fl}}$, and one concludes that the sheaf being represented is $\pi_{i-i}^s(\text{Map}_*(\mathbb{G}_m^{\wedge j}, L_{\mathbb{A}^1}^{\text{fl}} E)) \cong \pi_{i-i+j\alpha}^{s, \mathbb{A}^1}(E)$. \square

Corollary 2.16. *For any $i, j \geq 0$, and any object E of $\mathbf{Spt}(\mathbf{Sm}_k)$, taking global sections gives*

$$\pi_i^s(E)(*) = [\Sigma^\infty S^i, E]$$

and

$$\pi_{i+j\alpha}^{s, \mathbb{A}^1}(E)(*) = [\Sigma^\infty S^{i+j\alpha}, E]_{\mathbb{A}^1}.$$

Proposition 2.17. *Suppose $\{E_n\}$ is a filtered system of objects in $\mathbf{Spt}(\mathbf{Sm}_k)$. Then the natural maps*

$$\text{colim}_n \pi_i^s(E_i) \rightarrow \pi_i^s(\text{colim}_n E_i)$$

and

$$\text{colim}_n \pi_{i+j\alpha}^{s, \mathbb{A}^1}(E_i) \rightarrow \pi_{i+j\alpha}^{s, \mathbb{A}^1}(\text{colim}_n E_i)$$

are isomorphisms.

Proof. These follow from Corollary 2.11 and the observations that taking colimits commute and that colimits of spectra are calculated termwise. \square

We will use the term *cofiber sequence* only in a limited sense: a cofiber sequence in a pointed model category \mathbf{M} is a sequence of maps $X \rightarrow Y \rightarrow Z$ such that $X \rightarrow Y$ is a cofibration of cofibrant spaces and Z is a categorical pushout of $* \leftarrow X \rightarrow Y$. A *fiber sequence* is dual.

The image of a cofiber sequence in $\text{ho } \mathbf{M}$ may also be called a cofiber sequence, as in [Hov99, Chapter 6]. The notion of *fiber sequence* is dual.

The derived functors of left-Quillen functors preserve cofiber sequences, and dually the derived functors of right-Quillen functors preserve fiber sequences.

Suppose \mathfrak{a} is a model structure on $\mathbf{sPre}(\mathbf{Sm}_k)_*$, obtained as a left Bousfield localization of the flasque- or injective-local model structure. Consider $\mathbf{Spt}(\mathbf{Sm}_k)$, endowed with the stable model structure derived from \mathfrak{a} , [Hov01, Section 3]. By [Hov01, Theorem 3.9], the homotopy category $\text{ho}_{\mathfrak{a}} \mathbf{Spt}(\mathbf{Sm}_k)$ is a triangulated category in the sense of [Hov99, Chapter 7.1]. Write $\pi_{i+j\alpha}^{s, \mathfrak{a}}(E)$ for the sheaf associated to the presheaf $\mathcal{U} \mapsto [S^i \wedge \mathbb{G}_m^{\wedge j} \wedge \mathcal{U}_+, E]_{s, \mathfrak{a}}$, where the set of maps is calculated in $\text{ho}_{\mathfrak{a}}(\mathbf{Spt}(\mathbf{Sm}_k))$. Then we have the following result as an immediate corollary of sheafifying Lemma 7.1.10 of [Hov99].

Proposition 2.18. *If $X \rightarrow Y \rightarrow Z$ is a cofiber sequence in $\text{ho}_{\mathfrak{a}}(\mathbf{Spt}(\mathbf{Sm}_k))$, then the induced sequence of homotopy sheaves*

$$\rightarrow \pi_{i+j\alpha}^{s, \mathfrak{a}}(X) \rightarrow \pi_{i+j\alpha}^{s, \mathfrak{a}}(Y) \rightarrow \pi_{i+j\alpha}^{s, \mathfrak{a}}(Z) \rightarrow \pi_{i-1+j\alpha}^{s, \mathfrak{a}}(X) \rightarrow$$

is an exact sequence of sheaves of abelian groups.

Examples include $\pi_i^s, \pi_{i+j\alpha}^{s, \mathbb{A}^1}$ as well as $\pi_i^{s, P}$ and $\pi_{i+j\alpha}^{s, P, \mathbb{A}^1}$ of Section 3.

2.5. \mathbb{A}^1 Unstable and Stable. We say that a spectrum E is \mathbb{A}^1 - n -connected if $\pi_i^{s, \mathbb{A}^1}(E) = 0$ for all $i \leq n$. From the above definition of π_i^{s, \mathbb{A}^1} , combined with the theorem of [Mor12] saying that $L_{\mathbb{A}^1}$ does not decrease the connectivity of connected objects, and that $L_{\mathbb{A}^1}$ commutes with Ω for simply-connected objects, we deduce the following lemma:

Lemma 2.19. *If X is an \mathbb{A}^1 - n -connected object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, then $\Sigma^\infty X$ is \mathbb{A}^1 - n -connected.*

Recall that a map $f : X \rightarrow Y$ of connected objects of $\mathbf{sPre}(\mathbf{Sm}_k)_*$ is said to be n -connected if the homotopy fiber is $(n-1)$ -connected, and \mathbb{A}^1 - n -connected if the \mathbb{A}^1 -homotopy fiber is \mathbb{A}^1 -($n-1$)-connected.

By use of [Mor12, Theorem 6.53, Lemma 6.54] and the \mathbb{A}^1 -connectivity theorem, we deduce that if $X \rightarrow Y$ is n -connected with $n \geq 1$ and if moreover $\pi_1(Y)$ is strongly \mathbb{A}^1 invariant, then $X \rightarrow Y$ is \mathbb{A}^1 - n -connected. These conditions hold when X is simply connected, or when $n \geq 2$ and X is \mathbb{A}^1 local.

The following result is due to Asok–Fasel, [AF13].

Proposition 2.20 (The Blakers–Massey Theorem of Asok–Fasel). *Suppose $f : X \rightarrow Y$ is an \mathbb{A}^1 - n -connected map of connected objects in $\mathbf{sPre}(\mathbf{Sm}_k)_*$ and X is \mathbb{A}^1 - m -connected, with $n, m \geq 1$, then the morphism $\mathrm{hofib}_{\mathbb{A}^1} f \rightarrow \Omega L_{\mathbb{A}^1} \mathrm{hocofib} f$ is $m+n$ -connected.*

Proof. We rely on a homotopy excision result, a consequence of the Blakers–Massey theorem, that says that the result of this proposition holds in the setting of classical topology, [Whi12, Theorem 7.12].

We replace $f : X \rightarrow Y$ by an equivalent \mathbb{A}^1 -fibration of \mathbb{A}^1 -fibrant objects without changing the \mathbb{A}^1 homotopy type of $\mathrm{hofib}_{\mathbb{A}^1} f$ or of $\mathrm{hocofib} f$.

The \mathbb{A}^1 homotopy fiber of f therefore agrees with the ordinary fiber and therefore also with the simplicial homotopy fiber.

The classical homotopy excision result, applied at points, now says that the map $\mathrm{hofib} f \rightarrow \Omega \mathrm{hocofib} f$ is simplicially $(m+n)$ -connected. Since $m+n \geq 2$, and $\mathrm{hofib} f$ is \mathbb{A}^1 -local, it follows that $\pi_1(\Omega \mathrm{hocofib} f)$ is strongly \mathbb{A}^1 -invariant and then by [Mor12, Theorem 6.53, Lemma 6.54] it follows that

$$\mathrm{hofib}_{\mathbb{A}^1} f \simeq L_{\mathbb{A}^1} \mathrm{hofib}_{\mathbb{A}^1} f \rightarrow L_{\mathbb{A}^1} \Omega \mathrm{hocofib} f$$

is $(m+n)$ -connected.

The connectivity hypotheses imply that $\pi_1(Y) \cong \pi_1^{\mathbb{A}^1}(Y)$ is trivial, and therefore by the van Kampen theorem, that $\mathrm{hocofib} f$ is simply connected. This implies by [Mor12, Theorem 6.46] that $L_{\mathbb{A}^1} \Omega \mathrm{hocofib} f \simeq \Omega L_{\mathbb{A}^1} \mathrm{hocofib} f$. This completes the proof. \square

Corollary 2.21. *Suppose $f : X \rightarrow Y$ is a map of \mathbb{A}^1 simply connected objects in $\mathbf{sPre}(\mathbf{Sm}_k)_*$ such that the homotopy cofiber $\mathrm{hocofib} f$ is \mathbb{A}^1 contractible. Then f is an \mathbb{A}^1 weak equivalence.*

Proof. We show by induction that $\mathrm{hofib}_{\mathbb{A}^1} f$ is arbitrarily highly connected. Since X and Y are simply connected, $\mathrm{hofib}_{\mathbb{A}^1} f$ is 0-connected, so f is 1-connected.

Suppose we know that $\mathrm{hofib}_{\mathbb{A}^1} f$ is d -connected, then applying Proposition 2.20 with $n = d+1$ and $m = 1$, we deduce that $\mathrm{hofib}_{\mathbb{A}^1} f \rightarrow \Omega \mathrm{hocofib} f \simeq *$ is \mathbb{A}^1 -($d+2$)-connected, so that $\pi_{d+1}(\mathrm{hofib}_{\mathbb{A}^1} f)$ is trivial. \square

Corollary 2.22. *Suppose $f : X \rightarrow Y$ is a map of \mathbb{A}^1 simply connected objects in $\mathbf{sPre}(\mathbf{Sm}_k)_*$ such that $\Sigma^\infty f : \Sigma^\infty X \rightarrow \Sigma^\infty Y$ is an \mathbb{A}^1 -weak equivalence, then f is an \mathbb{A}^1 weak equivalence.*

Proof. We may replace f by a fibration of \mathbb{A}^1 -fibrant objects.

The map f is necessarily 1-connected, and from the proposition we deduce that $\pi_1(\text{hocofib } f) \simeq \pi_0(\text{hofib } f)$, which is trivial. Since Σ^∞ is a left Quillen functor, it preserves cofiber sequences in the derived category, and we deduce that $\Sigma^\infty \text{hocofib } f$ is \mathbb{A}^1 contractible. Since $\text{hocofib } f$ is simply connected, the \mathbb{A}^1 Hurewicz theorem implies that $\text{hocofib } f$ is \mathbb{A}^1 contractible.

An appeal to Corollary 2.21 now completes the argument. \square

2.6. Points. The site $\mathbf{Sh}_{\text{Nis}}(\mathbf{Sm}_k)$ is well known to have enough points. Let Q be a conservative set of points of $\mathbf{Sh}_{\text{Nis}}(\mathbf{Sm}_k)$. For each element $q \in Q$, there is an adjunction of categories

$$q^* : \mathbf{Sh}_{\text{Nis}}(\mathbf{Sm}_k) \rightleftarrows \mathbf{Set} : q_*,$$

where q^* , as well as preserving all colimits, preserves finite limits.

There is a Quillen adjunction

$$q^* : \mathbf{sPre}(\mathbf{Sm}_k) \rightleftarrows \mathbf{sSet} : q_*$$

from the injective local model structure on $\mathbf{sPre}(\mathbf{Sm}_k)$ to the usual model structure on \mathbf{sSet} . This extends in the obvious way to the pointed model categories, and to the categories of spectra

$$q^* : \mathbf{Spt}(\mathbf{Sm}_k) \rightleftarrows \mathbf{Spt} : q_*.$$

For an object X of $\mathbf{sPre}(\mathbf{Sm}_k)$, there is, by reference to 2.5, an isomorphism $q^* \pi_0(X) \cong \pi_0(q^* X)$. It is also the case that $p^* \Omega^i(L_{\text{Nis}} X) \simeq \Omega^i(\text{Ex}^\infty p^* X)$. This gives us the following proposition

Proposition 2.23. *If X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$ and i is a positive integer and q a point of $\mathbf{Sh}_{\text{Nis}}(\mathbf{Sm}_k)$, then there is an isomorphism of groups $\pi_i(|q^* X|) \cong q^* \pi_i(X)$.*

Corollary 2.24. *If X is an object of $\mathbf{Spt}(\mathbf{Sm}_k)$, and if i is an integer, then there is an isomorphism of abelian groups $\pi_i^s(q^* X) \cong q^* \pi_i^s(X)$.*

These facts are special cases of results concerning ∞ -topoi, [Lur09, 6.5.1.4]. They are well-known, see for instance [Mor05, 2.2 p14], but seldom stated.

3. LOCALIZATION

Let P denote a nonempty set of prime ideals of \mathbb{Z} , and $P' = \bigcap_{(p) \in P} (\mathbb{Z} \setminus (p))$ the set of integers not lying in any of these ideals. We write \mathbb{Z}_P for the localization $(P')^{-1} \mathbb{Z}$, and $\mathbb{Z}_{(p)}$ in the case where $P = \{(p)\}$ consists of a single ideal. Following [CP93], where the following is carried out in the category of CW complexes, we define $S_\tau^1 = S^1$, a Kan complex equivalent to $\Delta^1 / \partial \Delta^1$, and $S_\tau^n = S_\tau^1 \wedge (\Delta^{k-1} / \partial \Delta^{k-1})$. For any integer n , define $\rho_n^1 : S_\tau^1 \rightarrow S_\tau^1$ to be the usual degree- n self-map of S^1 , and extend this to maps $\rho_n^k : S_\tau^k \rightarrow S_\tau^k$ by $\rho_n^k = \rho_n^1 \wedge \text{id}$. Define T_P to be the set of maps

$$T_P = \{\rho_n^k : k \geq 1, n \in P'\}.$$

For each map $\rho_n^k : S_\tau^k \rightarrow S_\tau^k$ in T_P and each object U of \mathbf{Sm}_k , we may define a self-map $\rho_n^k \times \text{id}_U$ of $S_\tau^k \times U$. Denote the set of such maps by T'_P .

The local injective and flasque model structures on $\mathbf{sPre}(\mathbf{Sm}_k)$ are cellular in the sense of Hirschhorn, [Hir03]; a proof for the injective case appears in [Hor06, Lemma 1.5] and the flasque case is treated in [Isa05]. We may therefore apply the general machinery of [Hir03] and left-Bousfield-localize $\mathbf{sPre}(\mathbf{Sm}_k)$ at the set T'_P . We call the resulting model structures P -local, and if $P = \{(p)\}$ we call the resulting model structures

p -local. Write L_P for the functorial fibrant replacement functor in each model category. In the case where $P = \{(p)\}$, we may write $L_{(p)}$.

The localization of the usual model structure on \mathbf{sSet} with respect to the set T_P of maps is a form of P -local model structure on \mathbf{sSet} , we refer the reader to [CP93], especially [CP93, Section 8], for the comparisons between different P -localizations in classical topology and for a discussion of non-nilpotent objects. For nilpotent objects, the various P -localization functors agree up to weak equivalence.

Lemma 3.1. *With notation as above, if s is a point of $\mathbf{Sh}_{\text{Nis}}(\mathbf{Sm}_k)$, the adjunctions*

$$s^* : \mathbf{sPre}(\mathbf{Sm}_k) \rightleftarrows \mathbf{sSet} : s_*$$

and

$$s^* : \mathbf{sPre}(\mathbf{Sm}_k)_* \rightleftarrows \mathbf{sSet}_* : s_*$$

are monoidal Quillen adjunctions between the P -local model categories, where $\mathbf{sPre}(\mathbf{Sm}_k)$ and $\mathbf{sPre}(\mathbf{Sm}_k)_*$ may be given either the flasque or the injective model structure.

Proof. It is sufficient to prove the unpointed cases, the pointed follow immediately. The proofs in the flasque and injective cases are the same.

Following [Hir03, Theorem 3.3.20], the adjoint pair

$$s^* : \mathbf{sPre}(\mathbf{Sm}_k) \rightleftarrows \mathbf{sSet} : s_*$$

is a Quillen adjunction between the P -local model structure on the left and the model structure on \mathbf{sSet} obtained by localization at the set of maps

$$s^*(\rho_n^k \times \text{id}_U) : s^*(S_\tau^k \times U) \rightarrow s^*(S_\tau^k \times U)$$

where $\rho_n^k \in T_P$. Denote this set of maps by $s^*T'_P$. It will suffice to show that localization of \mathbf{sSet} at $s^*T'_P$ agrees with localization of \mathbf{sSet} at T_P .

Since evaluation at s^* commutes with fiber products, the maps of $s^*T'_P$ are of the form $\rho_n^k \times \text{id}_{s^*U}$, and setting $U = *$, we see that $T_P \subset s^*T'_P$. The maps of $s^*T'_P$ are, moreover, weak equivalences in the localization of \mathbf{sSet} at T_P . It follows that the localization of \mathbf{sSet} at $s^*T'_P$ is simply the ordinary P -localization of \mathbf{sSet} . \square

We note in addition that the model categories appearing above are simplicial model categories, and the adjunctions appearing are adjunctions of simplicial model categories in the sense of [Hov99, Chapter 4.2].

We continue to work principally in the injective local not-localized-at- P model structures, but write $A \simeq_P B$ to indicate that A is weakly equivalent to B in the P -local structure, or equivalently that $L_P A \simeq L_P B$. The notation $A \simeq_{(P)} B$ will be used where appropriate. We will use the flasque model structures only when dealing with spectra.

In this section we will occasionally write groups $\pi_i(X)$ in multiplicative notation even when the groups are abelian. The n -th power map of a group G will be the map $x \rightarrow x^n$, which is necessarily a homomorphism if G is abelian, and is preserved by group homomorphisms in any case. If P is a set of primes, then a group G is said to be P -local if the n -th power map is a bijection on G whenever n is not divisible by any of the primes in P . We will say that a presheaf of groups is P -local if all groups of sections are P -local, and a sheaf of groups is P -local if the appropriate n -th power maps are isomorphisms of sheaves of sets.

Proposition 3.2. *If X is a connected object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, and P is a set of primes, then the sheaves $\pi_i(L_P X)$ are P -local sheaves of groups.*

Proof. It suffices to show that the presheaves

$$\mathcal{U} \mapsto \pi_i(L_P X(\mathcal{U}))$$

are P -local, the result for the associated sheaves is then an exercise in sheafification.

Let n be an integer not divisible by any of the primes of P , let $i \geq 1$, and let \mathcal{U} be an object of \mathbf{Sm}_k . We wish to show that the n -th power map on $\pi_i(L_P X(\mathcal{U}))$ is a bijection, but this is the map induced by $\rho_n^i \times \text{id}_{\mathcal{U}}$ on $\pi_0(\text{SMap}_*(S_\tau^i \times \mathcal{U}, L_P X))$. Since $L_P X$ is P -local and $\rho_n^i \times \text{id}_{\mathcal{U}}$ is in T'_P , this map is a bijection. \square

Lemma 3.3. *Let X be an object of $\mathbf{sPre}(\mathbf{Sm}_k)$, let s be a point of $\mathbf{Sh}_{\text{Nis}}(\mathbf{Sm}_k)$ and let P be a set of primes. Then $s^*L_P X \simeq L_P s^*X$.*

Proof. We first claim that $s^*L_P X$ is P -local. Since it is fibrant, it suffices to show that if ρ_n^k is an element of T_P , then the induced map

$$(\rho_n^k)_* : \text{SMap}(S_\tau^k, s^*L_P X) \rightarrow \text{SMap}(S_\tau^k, s^*L_P X)$$

is a weak equivalence. If $\{\mathcal{U}_i\}$ is a system of neighbourhoods for s^* then there is a succession of natural isomorphisms

$$\begin{aligned} \text{SMap}(S_\tau^k, s^*L_P X) &\cong \text{SMap}(S_\tau^k, \text{colim}_{\mathcal{U}}(L_P X)(\mathcal{U})) \\ &\cong \text{colim}_{\mathcal{U}} \text{SMap}(S_\tau^k, (L_P X)(\mathcal{U})) \quad \text{since } S_\tau^k \text{ is compact,} \\ &\cong \text{colim}_{\mathcal{U}} \text{SMap}(S_\tau^k \times \mathcal{U}, L_P X) \end{aligned}$$

and ρ_n^k induces a weak equivalence on the spaces $\text{SMap}(S_\tau^k \times \mathcal{U}, L_P X)$ since $L_P X$ is P -local.

The functor s^* preserves trivial cofibrations, and therefore the map $s^*X \rightarrow s^*L_P X$ is a trivial cofibration the target of which is fibrant in the P -local model structure on \mathbf{sSet} . Therefore $s^*L_P X$ is weakly equivalent in the ordinary model structure on \mathbf{sSet} to any other P -fibrant-replacement for s^*X , notably to $L_P s^*X$, which is what was claimed. \square

Proposition 3.4. *Let X be a fibrant object of $\mathbf{sPre}(\mathbf{Sm}_k)$, let S be a conservative set of points of $\mathbf{Sh}_{\text{Nis}}(\mathbf{Sm}_k)$ and let P be a set of primes. Then X is P -local if and only if s^*X is P -local for all $s^* \in S$.*

Proof. For the ‘only-if’ direction, note that $X \simeq L_P X$ and then Lemma 3.3 directly asserts that the stalks, $s^*X \simeq L_P s^*X$, so they too are P -local.

For the ‘if’ direction we argue as follows. The space X is P -local if and only if X is fibrant and $X \rightarrow L_P X$ is a local weak equivalence. This is the case if and only if $s^*X \rightarrow s^*L_P X$ is a weak equivalence for all $s^* \in S$, which, by Lemma 3.3, is the case if and only if $s^*X \rightarrow L_P s^*X$ is a weak equivalence for all $s^* \in S$, and since s^*X is fibrant, this is the same as saying that s^*X is P -local in \mathbf{sSet} . \square

Definition 3.5. An object X of $\mathbf{sPre}(\mathbf{Sm}_k)_*$ is said to be *simple* if the action of $\pi_1(X)$ on $\pi_i(X)$ is trivial for all $i \geq 1$.

In particular, if X is simple then the sheaf $\pi_1(X)$ is a sheaf of abelian groups which acts trivially on $\pi_i(X)$ for all $i \geq 2$. A simply-connected object is simple, as is an object with an H -space structure.

Proposition 3.6. *Let X be a connected, simple object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, then the natural maps $\mathbb{Z}_P \otimes \pi_i(X) \rightarrow \pi_i(L_P X)$ are isomorphisms.*

Proof. Fix a point s of $\mathbf{Sh}_{\text{Nis}}(\mathbf{Sm}_k)$. By Lemma 3.3, there are isomorphisms

$$s^*\pi_i(L_P X) \cong \pi_i(s^*L_P X) \cong \pi_i(L_P s^*X).$$

By the results of [BK72], the last group is isomorphic to

$$\pi_i(s^*X) \otimes_{\mathbb{Z}} \mathbb{Z}_P \cong s^*(\pi_i(X) \otimes_{\mathbb{Z}} \mathbb{Z}_P)$$

which proves the proposition. \square

Lemma 3.7. *If X is a simply connected object of $\mathbf{sPre}(\mathbf{Sm}_k)$ and P a set of prime numbers, then $L_P(S^1 \wedge X) \simeq S^1 \wedge L_P X$, and $\Omega L_P X \simeq L_P \Omega L_{\text{Nis}} X$.*

Proof. For a pointed simplicial set X there is a map $S^1 \wedge X \rightarrow S^1 \wedge L_P X$ which induces P -localization on homology, and therefore there is a weak equivalence $L_P(S^1 \wedge X) \simeq S^1 \wedge L_P X$. This is promoted to the setting of simply connected objects in $\mathbf{sPre}(\mathbf{Sm}_k)_+$ by arguing at points.

A similar argument applies to ΩX , indeed without any connectivity assumptions, using homotopy in place of homology. \square

Proposition 3.8. *If X and Y are objects in $\mathbf{sPre}(\mathbf{Sm}_k)_*$ and P is a set of primes, then $L_P(X \times Y) \simeq L_P X \times L_P Y$.*

Proof. The object $L_P X \times L_P Y$ is P -locally weakly equivalent to $X \times Y$, and therefore to $L_P(X \times Y)$, by Lemma 2.1. Since $L_P X \times L_P Y$ is P -locally fibrant, the result follows. \square

3.1. P and \mathbb{A}^1 Localization.

Proposition 3.9. *If X is a connected object of $\mathbf{sPre}(\mathbf{Sm}_k)$ such that X is \mathbb{A}^1 -local and $\pi_1(X)$ is abelian, then $L_P X$ is again \mathbb{A}^1 local.*

Proof. Under the hypotheses, it suffices to check that the sheaves $\pi_i(X) \otimes_{\mathbb{Z}} \mathbb{Z}_P$ are strictly \mathbb{A}^1 -invariant, [Mor12, Chapter 6], but this follows immediately since the functor $\cdot \otimes_{\mathbb{Z}} \mathbb{Z}_P$ is exact. \square

In the sequel, we consider only the composite localization $L_P L_{\mathbb{A}^1} X$, and not the reverse. The proposition says that, under connectivity hypotheses, $L_P L_{\mathbb{A}^1} X$ is both \mathbb{A}^1 and P -local.

If X is a connected H -space in $\mathbf{sPre}(\mathbf{Sm}_k)_*$, then it is possible to define self maps

$$\times n : X \xrightarrow{\Delta} X^{\mu(\mu(\dots\mu))} X$$

by composing the n -fold diagonal and an iterated multiplication map. The map $\times n$ represents a class in $[X, X]$, which we also denote $\times n$ in an abuse of notation.

Proposition 3.10. *If X is a connected H -space in $\mathbf{sPre}(\mathbf{Sm}_k)_*$ and P is a set of primes, then $L_P X$ is again a connected H -space, and the map $X \rightarrow L_P X$ is a weak equivalence if and only if $\times n \in [X, X]$ is invertible for all n not divisible by the primes of P .*

Proof. The object $L_P X$ carries an H -space structure since $L_P(X \times X) \simeq L_P X \times L_P X$, see Proposition 3.8.

An Eckmann–Hilton argument implies that X is simple, that is the action of $\pi_1(X)$ on $\pi_i(X)$ is trivial for all i , and moreover $\times n$ induces multiplication by n on all homotopy sheaves. The result follows. \square

Proposition 3.11. *Suppose X is a connected object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, and further that X is equipped with an H -space structure. Then $L_{\mathbb{A}^1} L_P X \simeq L_P L_{\mathbb{A}^1} X$, where the localizations are carried out with respect to either the local or the flasque model structure on $\mathbf{sPre}(\mathbf{Sm}_k)_*$.*

Proof. We give the proof in the local case, the flasque is the same *mutatis mutandis*.

Starting with the Quillen adjunction from the injective local model structure on $\mathbf{sPre}(\mathbf{Sm}_k)_*$ to the \mathbb{A}^1 local, we obtain a commutative diagram of model structures, where the maps indicated are left Quillen adjoints:

$$(8) \quad \begin{array}{ccc} \text{Local} & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \text{P-local} & \longrightarrow & \text{P-}\mathbb{A}^1 \end{array}$$

where the $\text{P-}\mathbb{A}^1$ model structure is the P-localization in the evident sense of the \mathbb{A}^1 model structure.

We claim that for a connected H-space object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, the maps $X \rightarrow L_{\mathbb{A}^1} L_P X$ and $X \rightarrow L_P L_{\mathbb{A}^1} X$ are both fibrant replacements in the $\text{P-}\mathbb{A}^1$ -model structure, and therefore that $L_{\mathbb{A}^1} L_P X \simeq L_P L_{\mathbb{A}^1} X$ in the original model structure.

The lynchpin of the following argument is the observation, by reference to [Hir03, Proposition 3.4.1], an object W of $\mathbf{sPre}(\mathbf{Sm}_k)$ is $\text{P-}\mathbb{A}^1$ -local if it satisfies the following three conditions:

- (1) W is fibrant in the injective model structure on $\mathbf{sPre}(\mathbf{Sm}_k)$.
- (2) For any object U of \mathbf{Sm}_k , the maps

$$\text{SMap}(U, W) \rightarrow \text{SMap}(U \times \mathbb{A}^1, W)$$

of simplicial mapping objects are weak equivalences.

- (3) For any ρ_n^k where $k \geq 1$ and n is not divisible by a prime in P , the maps induced by ρ_n^k

$$\text{SMap}(S_\tau^k, W) \rightarrow \text{SMap}(S_\tau^k, W).$$

The object $L_P L_{\mathbb{A}^1} X$ is both \mathbb{A}^1 -fibrant, since L_P preserves \mathbb{A}^1 -fibrancy for H-space objects, and P-locally fibrant, and it is therefore a P-local object in the \mathbb{A}^1 model structure. By reference to [Hir03, Proposition 3.4.1], it is fibrant in the $\text{P-}\mathbb{A}^1$ -model structure. Since $X \rightarrow L_{\mathbb{A}^1} X$ is an \mathbb{A}^1 weak equivalence, it is *a fortiori* a $\text{P-}\mathbb{A}^1$ -weak equivalence, and therefore $X \rightarrow L_{\mathbb{A}^1} X \rightarrow L_P L_{\mathbb{A}^1} X$ is a $\text{P-}\mathbb{A}^1$ -weak equivalence, and therefore a fibrant replacement.

Similarly, $L_{\mathbb{A}^1} L_P X$ is \mathbb{A}^1 -fibrant and, since $L_{\mathbb{A}^1}$ preserves P-local fibrancy for simple objects, P-locally fibrant. Moreover $X \rightarrow L_P X$ is a P-local weak equivalence, and therefore a $\text{P-}\mathbb{A}^1$ weak equivalence, and consequently $X \rightarrow L_P X \rightarrow L_{\mathbb{A}^1} L_P X$ is a $\text{P-}\mathbb{A}^1$ fibrant replacement. \square

We recall from [Mor12, Chapter 2], that there is a construction on presheaves of groups, \mathcal{G} , given by

$$\mathcal{G}_{-1} : \mathcal{U} \mapsto \ker(\mathcal{G}(\mathbb{G}_m \times X) \xrightarrow{\text{ev}(1)} \mathcal{G}(X))$$

where $\text{ev}(1)$ is evaluation at 1 in \mathbb{G}_m . Equivalently, \mathcal{G}_{-1} is the kernel of the map of group sheaves

$$\text{Map}(\mathbb{G}_m, \mathcal{G}) \rightarrow \text{Map}(*, \mathcal{G}) \cong \mathcal{G}.$$

The assignment $\mathcal{G} \mapsto \mathcal{G}_{-1}$ is functorial, and sends sheaves to sheaves. The j -fold iterate of the ‘ -1 ’ functor applied to \mathcal{G} is denoted by \mathcal{G}_{-j} .

The result [Mor12, Theorem 6.13] says that if X is a connected object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, then $\pi_{i+j\alpha}^{\mathbb{A}^1}(X) = \pi_i^{\mathbb{A}^1}(X)_{-j}$. Recall that $\pi_{i+j\alpha}^{\mathbb{A}^1}(X)$ is notation for $\pi_i(\text{Map}(\mathbb{G}_m^{\wedge j}, L_{\mathbb{A}^1} X))$.

Proposition 3.12. *If \mathcal{G} is an abelian sheaf of groups, then \mathcal{G}_{-1} is also abelian and there is a natural isomorphism $(R \otimes \mathcal{G})_{-1} \cong R \otimes \mathcal{G}_{-1}$.*

Proof. The abelian property of \mathcal{G}_{-1} follows immediately from the definition.

For any object U of \mathbf{Sm}_k , we have a natural commutative diagram of left-exact sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & (R \otimes \mathcal{G})_{-1}(U) & \longrightarrow & (R \otimes \mathcal{G})(\mathbb{G}_m \times U) & \longrightarrow & (R \otimes \mathcal{G})(U) \\ & & \downarrow f & & \parallel & & \parallel \\ 1 & \longrightarrow & R \otimes \mathcal{G}_{-1}(U) & \longrightarrow & R \otimes (\mathcal{G}(\mathbb{G}_m \times U)) & \longrightarrow & R \otimes (\mathcal{G}(U)), \end{array}$$

from which the natural isomorphism $(R \otimes \mathcal{G})_{-1} \cong R \otimes \mathcal{G}_{-1}$ follows. \square

Proposition 3.13. *If j is a nonnegative integer, X is a simply connected object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, and P is a set of primes, then there is a natural isomorphism $\mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_P L_{\mathbb{A}^1} X) \rightarrow L_P \mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_{\mathbb{A}^1} X)$ in $\mathrm{hoNis} \mathbf{sPre}(\mathbf{Sm}_k)_*$.*

Proof. Each of the two spaces in question is equipped with a natural map to $L_P \mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_P L_{\mathbb{A}^1} X)$. It suffices to show that each of these maps is a simplicial weak equivalence.

By Proposition 3.9, the space $L_P L_{\mathbb{A}^1} X$ is \mathbb{A}^1 -local. By the unstable \mathbb{A}^1 connectivity theorem, [Mor12, Theorem 6.38], it is also connected. As is shown in the proof of [Mor12, Theorem 6.13], the functor $\mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, \cdot)$, preserves the subcategory of connected, \mathbb{A}^1 -local objects in $\mathbf{sPre}(\mathbf{Sm}_k)_*$.

Let Y denote either $L_{\mathbb{A}^1} X$ or $L_P L_{\mathbb{A}^1} X$, both of which are \mathbb{A}^1 -local and connected. Then, for any $i \geq 1$, the homotopy sheaf $\pi_i(\mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_P Y))$ is naturally isomorphic to

$$\pi_i(L_P Y)_{-j} \cong \pi_i^{P, \mathbb{A}^1}(Y)_{-j} \cong \pi_i^{\mathbb{A}^1}(Y)_{-j} \otimes_{\mathbb{Z}} \mathbb{Z}_P \cong \pi_i(\mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, Y)) \otimes_{\mathbb{Z}} \mathbb{Z}_P \cong \pi_i(L_P \mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, Y)).$$

In particular the spaces $\mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_P L_{\mathbb{A}^1} X)$ and $L_P \mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_{\mathbb{A}^1} X)$ are both weakly equivalent to

$$L_P \mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_P L_{\mathbb{A}^1} X),$$

as required. \square

Definition 3.14. For an object X of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, and nonnegative integers i and j , the notation $\pi_{i+j\alpha}^{P, \mathbb{A}^1}(X)$ is used to denote $\pi_i(\mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_P L_{\mathbb{A}^1} X))$.

Proposition 3.15. *If i, j are nonnegative integers, $L_{\mathbb{A}^1} X$ is a simply connected, \mathbb{A}^1 local object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, and P a set of primes, then there are natural isomorphisms*

$$\pi_{i+j\alpha}^{P, \mathbb{A}^1}(X) \cong \pi_i^{P, \mathbb{A}^1}(X)_{-j} \cong \pi_i^{\mathbb{A}^1}(X)_{-j} \otimes_{\mathbb{Z}} \mathbb{Z}_P \cong \pi_{i+j\alpha}^{\mathbb{A}^1}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_P.$$

Proof. The sheaf $\pi_{i+j\alpha}^{P, \mathbb{A}^1}(X)$ is isomorphic to $\pi_{i+j\alpha}^{\mathbb{A}^1}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_P$ by Proposition 3.13. This is isomorphic to $\pi_i^{\mathbb{A}^1}(X)_{-j} \otimes_{\mathbb{Z}} \mathbb{Z}_P \cong \pi_i^{P, \mathbb{A}^1}(X)_{-j}$ as required. \square

Proposition 3.16. *If P is a set of primes and if X, Y and Z are simply connected objects of $\mathbf{sPre}(\mathbf{Sm}_k)_*$ such that $X \rightarrow Y \rightarrow Z$ is a P - \mathbb{A}^1 -fiber sequence up to homotopy, and if j is a nonnegative integer, then there is a natural long exact sequence*

$$\cdots \rightarrow \pi_{i+j\alpha}^{P, \mathbb{A}^1}(X) \rightarrow \pi_{i+j\alpha}^{P, \mathbb{A}^1}(Y) \rightarrow \pi_{i+j\alpha}^{P, \mathbb{A}^1}(Z) \rightarrow \pi_{i-1+j\alpha}^{P, \mathbb{A}^1}(X) \rightarrow \cdots$$

Proof. The hypothesis implies that $L_P L_{\mathbb{A}^1} X \rightarrow L_P L_{\mathbb{A}^1} Y \rightarrow L_P L_{\mathbb{A}^1} Z$ is a simplicial fiber sequence up to homotopy. Since the objects involved are fibrant, applying the functor $\mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, \cdot)$ yields another fiber sequence up to homotopy:

$$\mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_P L_{\mathbb{A}^1} X) \rightarrow \mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_P L_{\mathbb{A}^1} Y) \rightarrow \mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_P L_{\mathbb{A}^1} Z)$$

which, by Proposition 3.13, is weakly equivalent to

$$L_P \mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_{\mathbb{A}^1} X) \rightarrow L_P \mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_{\mathbb{A}^1} Y) \rightarrow L_P \mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_{\mathbb{A}^1} Z).$$

The long exact sequence of homotopy sheaves is

$$\begin{aligned} & \rightarrow \pi_i(\mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_{\mathbb{A}^1} X)) \otimes_{\mathbb{Z}} \mathbb{Z}_P \rightarrow \pi_i(\mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_{\mathbb{A}^1} Y)) \otimes_{\mathbb{Z}} \mathbb{Z}_P \rightarrow \\ & \rightarrow \pi_i(\mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_{\mathbb{A}^1} Z)) \otimes_{\mathbb{Z}} \mathbb{Z}_P \rightarrow \pi_{i-1}(\mathrm{Map}_*(\mathbb{G}_m^{\wedge j}, L_{\mathbb{A}^1} X)) \otimes_{\mathbb{Z}} \mathbb{Z}_P \rightarrow, \end{aligned}$$

and, by Proposition 3.12 and 3.15, this is naturally isomorphic to

$$\rightarrow \pi_{i+j\alpha}^{P, \mathbb{A}^1}(X) \rightarrow \pi_{i+j\alpha}^{P, \mathbb{A}^1}(Y) \rightarrow \pi_{i+j\alpha}^{P, \mathbb{A}^1}(Z) \rightarrow \pi_{i-1+j\alpha}^{P, \mathbb{A}^1}(X) \rightarrow$$

as required. \square

3.2. P Localization of Spectra. Throughout this section, the underlying model structure on $\mathbf{sPre}(\mathbf{Sm}_k)_*$ is taken to be the flasque, rather than the injective.

One can construct a P-local model category of presheaves of spectra, following [Hov01], as the S^1 -stable model category on the P-local flasque model structure on $\mathbf{sPre}(\mathbf{Sm}_k)_*$.

Lemma 3.17. *The adjunction*

$$\Sigma^\infty : \mathbf{sPre}(\mathbf{Sm}_k)_* \rightleftarrows \mathbf{Spt}(\mathbf{Sm}_k) : \mathrm{Ev}_0$$

is a Quillen adjunction between the P-local model categories.

Proof. This is implicit in [Hov01], being the combination of Proposition 1.16 and the definition of the stable model structure as a localization of the level model structure on spectra. \square

Explicitly, the fibrant-replacement functor, L_P , in $\mathbf{Spt}(\mathbf{Sm}_k)$ with the P-local model structure is given by

$$(L_P E)_i = \mathrm{colim}_{k \rightarrow \infty} \mathrm{Map}_*(S^k, L_P E_{i+k}).$$

With the P-local model structures and the smash product, the category $\mathbf{Spt}(\mathbf{Sm}_k)$ is a $\mathbf{sPre}(\mathbf{Sm}_k)_*$ -model category, in the sense of [Hov99, Chapter 4.2]

Lemma 3.18. *If s is a point of $\mathbf{Sh}_{\mathrm{Nis}}(\mathbf{Sm}_k)$, the adjunction*

$$s^* : \mathbf{Spt}(\mathbf{Sm}_k) \rightleftarrows \mathbf{Spt} : s_*$$

is a Quillen adjunction between the P-local model categories.

Corollary 3.19. *For any object X of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, and any set P of primes, there is a stable weak equivalence:*

$$\Sigma^\infty L_P X \rightarrow L_P \Sigma^\infty X.$$

A spectrum E is said to be *P-local* if it is fibrant and the map $E \rightarrow L_P E$ is a stable weak equivalence. Since it is possible to check stable weak equivalence of spectra at points, we deduce the following by arguing at points.

Proposition 3.20. *A spectrum E is P-local if and only if it is fibrant and the maps*

$$E \xrightarrow{n} E$$

are weak equivalences for all n not divisible by the primes in P .

Proposition 3.21. *A spectrum E is P-local if and only if it is fibrant and the localization maps $\pi_i^s(E) \rightarrow \pi_i^s(E) \otimes_{\mathbb{Z}} \mathbb{Z}_P$ are isomorphisms for all i .*

3.3. P - and \mathbb{A}^1 -Localization of Spectra. We begin with a commutative diagram of model structures on the category $\mathbf{Spt}(\mathbf{Sm}_k)$, which is the application of [Hov01] to the flasque version of diagram (8):

$$\begin{array}{ccc} \text{Stable} & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ P\text{-Stable} & \longrightarrow & P\text{-}\mathbb{A}^1\text{-Stable.} \end{array}$$

Fibrant replacements in the \mathbb{A}^1 or P -local model structures are effected by replacing E by the spectrum that has level i given by

$$\operatorname{colim}_k \Omega^k L_{\mathbb{A}^1} E_{i+k}$$

or

$$\operatorname{colim}_k \Omega^k L_P E_{i+k}$$

respectively, L_P and $L_{\mathbb{A}^1}$ being taken in the flasque model structures. The stable fibrant replacements are denoted $L_P E$ and $L_{\mathbb{A}^1} E$.

Lemma 3.22. *The classes of P -locally flasque fibrant and $P\text{-}\mathbb{A}^1$ -locally flasque fibrant objects in $\mathbf{sPre}(\mathbf{Sm}_k)$ and $\mathbf{sPre}(\mathbf{Sm}_k)_*$ are closed under filtered colimits.*

Proof. Since an object is $P\text{-}\mathbb{A}^1$ -locally flasque fibrant if and only if it is both P - and \mathbb{A}^1 -locally flasque fibrant, it suffices to prove the case of P -locally flasque fibrant objects.

Suppose X_k is a filtered diagram of P -locally flasque fibrant objects, then $\operatorname{colim}_k X_k$ is flasque fibrant, by [Isa05]. We wish to show that for any $\rho_n^n \times \operatorname{id}_U : S_\tau^k \times U \rightarrow S_\tau^k \times U$ in T_P , the induced map

$$\operatorname{SMap}(S_\tau^k \times U, \operatorname{colim}_k X_k) \rightarrow \operatorname{SMap}(S_\tau^k \times U, \operatorname{colim}_k X_k)$$

is a weak equivalence. Since $S_\tau^k \times U$ is equivalent to a compact object, $S^k \times U$, of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, and since the X_k and $\operatorname{colim}_k X_k$ are all fibrant, and $S_\tau^k \times U$ and $S^k \times U$ are all cofibrant, the given map, we wish to show that the induced map

$$\operatorname{colim}_k \operatorname{SMap}(S_\tau^k \times U, X_k) \rightarrow \operatorname{colim}_k \operatorname{SMap}(S_\tau^k \times U, X_k)$$

is a weak equivalence of simplicial sets, but since the X_k are themselves P -locally fibrant, this is immediate. \square

Proposition 3.23. *For any object E of $\mathbf{Spt}(\mathbf{Sm}_k)$ there is a stable weak equivalence $L_P L_{\mathbb{A}^1} E \simeq L_{\mathbb{A}^1} L_P E$.*

Proof. The objects in question are levelwise fibrant for the flasque model structure. It suffices therefore to show that they are levelwise weakly equivalent for the flasque model structure. Since we are working the flasque model structure, filtered colimits of fibrant objects are again fibrant, and so we deduce the existence of weak equivalences

$$L_{\mathbb{A}^1} \operatorname{colim}_k X_k \simeq L_{\mathbb{A}^1} \operatorname{colim}_k L_{\mathbb{A}^1} X_k \simeq \operatorname{colim}_k L_{\mathbb{A}^1} X_k$$

and similarly for L_P .

We may assume that the spaces E_i appearing are all simply-connected H -spaces, and therefore $L_P L_{\mathbb{A}^1} E_i \simeq L_{\mathbb{A}^1} L_P E_i$. We then have

$$\operatorname{colim}_k \Omega^k L_{\mathbb{A}^1} \left(\operatorname{colim}_{k'} \Omega^{k'} L_P E_{k+k'+i} \right) \simeq \operatorname{colim}_k \Omega^k \left(\operatorname{colim}_{k'} \Omega^{k'} L_{\mathbb{A}^1} L_P E_{k+k'+i} \right)$$

which is symmetric in $L_{\mathbb{A}^1}$, L_P , up to weak equivalence, whence the result. \square

We therefore conflate $L_P L_{\mathbb{A}^1} E$ and $L_{\mathbb{A}^1} L_P E$, calling either the $P\text{-}\mathbb{A}^1$ -localization of E . We say that a map of spectra $f : E \rightarrow E'$ is a $P\text{-}\mathbb{A}^1$ -weak equivalence if $L_P L_{\mathbb{A}^1} f$ is a stable weak equivalence of spectra, or

equivalently if $L_P f$ is an \mathbb{A}^1 weak equivalence of spectra, or equivalently again if $L_{\mathbb{A}^1} f$ is a P -local equivalence of spectra. We write $\pi_i^{P, \mathbb{A}^1, s}(E)$ for the homotopy sheaves $\pi_i^s(L_P L_{\mathbb{A}^1} E)$.

Proposition 3.24. *If E is an object in $\mathbf{Spt}(\mathbf{Sm}_k)$, and P is a set of prime numbers then there is a natural isomorphism*

$$\pi_i^{s, P, \mathbb{A}^1}(E) \cong \pi_i^{s, \mathbb{A}^1}(E) \otimes_{\mathbb{Z}} \mathbb{Z}_P.$$

Proof. Immediate from the above. \square

Proposition 3.25. *If $f : E \rightarrow E'$ is a map in $\mathbf{Spt}(\mathbf{Sm}_k)$ and P is a set of prime numbers, then f is a P - \mathbb{A}^1 weak equivalence if and only if $\pi_i^{s, \mathbb{A}^1}(f) \otimes_{\mathbb{Z}} \mathbb{Z}_P$ is an isomorphism of abelian groups for all i .*

Proof. Immediate from the above. \square

Proposition 3.26. *Suppose $\{E_n\}$ is a filtered system of objects in $\mathbf{Spt}(\mathbf{Sm}_k)$ and P is a set of prime numbers, then the natural maps*

$$\operatorname{colim}_n \pi_i^{s, P}(E_n) \rightarrow \pi_i^{s, P}(\operatorname{colim}_n E_n)$$

and

$$\operatorname{colim}_n \pi_{i+j\alpha}^{s, P, \mathbb{A}^1}(E_n) \rightarrow \pi_{i+j\alpha}^{s, P, \mathbb{A}^1}(\operatorname{colim}_n E_n)$$

are isomorphisms

Proof. Immediate from the above and Proposition 2.17. \square

Proposition 3.27. *Suppose $f : X \rightarrow Y$ is a map of simply-connected objects $\mathbf{sPre}(\mathbf{Sm}_k)_*$ such that $\Sigma^\infty f$ is a P - \mathbb{A}^1 -stable weak equivalence. Then f is a P - \mathbb{A}^1 weak equivalence.*

Proof. The map $L_P L_{\mathbb{A}^1} f : L_P L_{\mathbb{A}^1} \Sigma^\infty X \rightarrow L_P L_{\mathbb{A}^1} \Sigma^\infty Y$ is a weak equivalence, and this map agrees in the stable homotopy category with $L_P L_{\mathbb{A}^1} \Sigma^\infty L_{\mathbb{A}^1} X \rightarrow L_P L_{\mathbb{A}^1} \Sigma^\infty L_{\mathbb{A}^1} Y$. We may commute L_P past $L_{\mathbb{A}^1}$ and past Σ^∞ , so that we conclude that $L_{\mathbb{A}^1} \Sigma^\infty L_P L_{\mathbb{A}^1} X \rightarrow L_{\mathbb{A}^1} \Sigma^\infty L_P L_{\mathbb{A}^1} Y$ is a weak equivalence. By Corollary 2.22, since $L_P L_{\mathbb{A}^1} X$ and $L_P L_{\mathbb{A}^1} Y$ are simply connected, we deduce that $L_{\mathbb{A}^1} L_P L_{\mathbb{A}^1} X \rightarrow L_{\mathbb{A}^1} L_P L_{\mathbb{A}^1} Y$ is a weak equivalence, and since $L_P L_{\mathbb{A}^1} X, L_P L_{\mathbb{A}^1} Y$ are already \mathbb{A}^1 -local by Proposition 3.9, the result follows. \square

4. THE GROTHENDIECK-WITT GROUP

4.1. The homotopy of spheres. Consider a motivic sphere $X = S^{n+q\alpha} = S^n \wedge \mathbb{G}_m^{\wedge q}$.

We make frequent use of the following result, which is a paraphrase of some results of [Mor12, Section 6.3]:

Lemma 4.1 (Morel). *If (n, q) and (n', q') are pairs of nonnegative integers, and if $n \geq 2$, then*

$$\pi_{n+q\alpha}^{\mathbb{A}^1}(S^{n'+q'\alpha}) = \begin{cases} 0 & \text{if } n < n'; \\ K_{q'-q}^{\text{MW}} & \text{if } n = n' \text{ and } q' > 0; \\ 0 & \text{if } n = n', q' > 0 \text{ and } q = 0; \\ \mathbb{Z} & \text{if } n = n' \text{ and } q = q' = 0. \end{cases}$$

The stable version of this result was known earlier, but may be deduced from the unstable.

Corollary 4.2. *If (n, q) and (n', q') are pairs of integers with q, q' nonnegative, then the sets of maps between $S^{n+q\alpha}$ and $S^{n'+q'\alpha}$ in the \mathbb{A}^1 homotopy category of S^1 -spectra take the form*

$$\pi_{n+q\alpha}^{s, \mathbb{A}^1}(S^{n'+q'\alpha}) = \begin{cases} 0 & \text{if } n < n'; \\ K_{q'-q}^{\text{MW}} & \text{if } n = n' \text{ and } q' > 0 \\ 0 & \text{if } n = n', q' > 0 \text{ and } q = 0; \\ \mathbb{Z} & \text{if } n = n' \text{ and } q = q' = 0. \end{cases}$$

We remark that K_0^{MW} is the sheaf of Grothendieck–Witt groups, also denoted GW . We observe that if $q > 0$, then, by Corollary 2.16:

$$[\Sigma^\infty S^{n+q\alpha}, \Sigma^\infty S^{n'+q'\alpha}]_{\mathbb{A}^1} = \text{GW}(*).$$

Proposition 4.3. *Suppose n, n', q, q' are integers such that n, q, q' are nonnegative and $q' \geq 2$. We have an identification*

$$\pi_{n+q\alpha}^{s, \mathbb{P}, \mathbb{A}^1}(S^{n'+q'\alpha}) = \begin{cases} 0 & \text{if } n < n', \\ K_{q'-q}^{\text{MW}} \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathbb{P}} & \text{if } n = n'. \end{cases}$$

Proof. This follows immediately from Section 4.1 and Proposition 3.24. \square

Remark 4.4. Since \mathbb{Z} is a subring of $K_0^{\text{MW}}(k) = \text{GW}(k)$, it follows that $\mathbb{Z}_{\mathbb{P}}$ is a subring of $\text{GW}(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathbb{P}}$.

4.2. Twist classes. For $a \in k^*$, following [Mor12, Chapter 3], we define $\langle a \rangle : S^{(1,0)} \wedge \mathbb{G}_m \rightarrow S^{(1,0)} \wedge \mathbb{G}_m$ to be the map induced by multiplication $a : \mathbb{G}_m \rightarrow \mathbb{G}_m$ by forming $a_+ : (\mathbb{G}_m)_+ \rightarrow \mathbb{G}_m$, suspending

$$S^{(1,0)} \wedge (a_+) : S^{(1,0)} \wedge (\mathbb{G}_m)_+ \cong S^{1,0} \vee S^{(1,0)} \wedge \mathbb{G}_m \rightarrow S^{(1,0)} \wedge \mathbb{G}_m$$

and letting $\langle a \rangle$ denote the restriction of this map to the $S^{(1,0)} \wedge \mathbb{G}_m$ summand.

Remark 4.5. The interchange of any two adjacent terms in $(S^{n+q\alpha})^{\wedge r} = S^{n+q\alpha} \wedge S^{n+q\alpha} \wedge \dots \wedge S^{n+q\alpha} \cong S^{rn+rq\alpha}$ represents the element

$$e_{n,q} = (-1)^{n+q} \langle -1 \rangle^q \in \pi_{rn+rq\alpha}^{\mathbb{A}^1}(S^{rn+rq\alpha}),$$

by [Mor12, Lemma 3.43]. Observe that $e_{n,q}^2 = 1$.

Much of the following work depends on showing a class $A + B\langle -1 \rangle$, where A and B are integers, is a unit in the ring $\text{GW}(k)$ or $\text{GW}(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}$.

We remind the reader that a field k is said to be *formally real* if -1 cannot be written as a sum of squares in k , [Lam73, Chapter VIII]. We also remind the reader that all our fields are assumed to have characteristic unequal to 2.

Proposition 4.6. *Suppose A, B are integers, and let R be a localization of \mathbb{Z} . Then $A + B\langle -1 \rangle$ is a unit in $\text{GW}(k) \otimes_{\mathbb{Z}} R$ if and only if one of the following conditions is met:*

- (1) $A + B$ and $A - B$ are units in R and k is formally real;
- (2) $A + B$ is a unit in R and the field k is not formally real.

Proof. We remark that the dimension homomorphism makes R into a split subring of $\text{GW}(k) \otimes_{\mathbb{Z}} R$.

We first approach the case where k is formally real.

Since $(A + B\langle -1 \rangle)(A + B\langle -1 \rangle) = A^2 - B^2 = (A + B)(A - B)$, the condition in (1) is sufficient.

We may embed k in a real closure $\phi : k \rightarrow k^r$. This embedding induces a ring homomorphism $\phi : GW(k) \otimes_{\mathbb{Z}} R \rightarrow GW(k^r) \otimes_{\mathbb{Z}} R = R \oplus R\langle -1 \rangle$, [Lam73, Proposition II.3.2]. Abstractly, the ring $GW(k^r) \otimes_{\mathbb{Z}} R$ is endowed with an automorphism $\langle -1 \rangle \mapsto -\langle -1 \rangle$, and if $\phi(A + B\langle -1 \rangle) = A + B\langle -1 \rangle$ is a unit, then so too is $A - B\langle -1 \rangle$, from which we deduce that their product, $(A + B\langle -1 \rangle)(A - B\langle -1 \rangle) = A^2 - B^2$ is a unit as well. But $A^2 - B^2$ is a unit if and only if $A + B$ and $A - B$ are units, showing that this condition is necessary and sufficient if k is formally real.

Suppose now that k is not formally real.

We may employ the dimension map $GW(k) \otimes_{\mathbb{Z}} R \rightarrow R$ to show that if $A + B\langle -1 \rangle$ is a unit, then necessarily $A + B$ is a unit, u .

We wish to show that this condition is also sufficient to imply $A + B\langle -1 \rangle$ is a unit. We may write $A = u - B$.

In the non-formally-real case the Witt group $W(k) = GW(k)/(1 + \langle -1 \rangle)$ is torsion, [Lam73, Theorem VIII.3.6]. The ideal $(1 + \langle -1 \rangle)$ consists of integer multiples of $1 + \langle -1 \rangle$, by basic quadratic-form theory, [Lam73, Definition II .1.3]. This implies that there exists some integers m, n , with $m > 0$, such that $m\langle -1 \rangle = n(1 + \langle -1 \rangle)$ in $GW(k)$, and the dimension homomorphism shows that $m = 2n$, so that $n(1 - \langle -1 \rangle) = 0$. It is the case that an element in $GW(k)$ is torsion if and only if it is nilpotent, [KRW72, Theorem 6], whereupon we deduce that $1 - \langle -1 \rangle$ is nilpotent in $GW(k)$ and therefore also in $GW(k) \otimes_{\mathbb{Z}} R$, so $A + B\langle -1 \rangle = u - B(1 - \langle -1 \rangle)$ is a unit in $GW(k) \otimes_{\mathbb{Z}} R$ as required. \square

We employ Proposition 4.6 via the following two corollaries.

Corollary 4.7. *Suppose m is a nonnegative integer and $e_{n,q} = (-1)^{n+q}\langle -1 \rangle^q$ is the twist class of the sphere $S^{n+q\alpha}$. Then the class $1 + m + me_{n,q}$ is a unit in $GW(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}$.*

Proof. There are, in general, four cases, $e_{n,q} = \pm 1$ and $e_{n,q} = \pm \langle -1 \rangle$; although it is possible that $\langle -1 \rangle = 1$ in $GW(k)$. The two cases $e_{n,q} = \pm 1$ are immediate.

For the other two cases, by the proposition, it suffices to check that one or both of $m + 1 + m = 2m + 1$ and $m + 1 - m = 1$ are units in $\mathbb{Z}_{(2)}$, which they both are. \square

Corollary 4.8. *Suppose m is a positive integer and $e_{n,q} = (-1)^{n+q}\langle -1 \rangle^q$ is the twist class of the sphere $S^{n+q\alpha}$. Then the class $1 + m + me_{n,q}$ is a unit in $GW(k)$ if and only if one of the following conditions holds*

- n is odd and q is even;
- $n + q$ is odd and k is not formally real.

Proof. The cases where q is even, so $e_{n,q} = \pm 1$, are easily dealt with and do not depend on the field. Assume therefore that q is odd.

Suppose k is formally real, then the Proposition says that $1 + m \pm m\langle -1 \rangle$ is a unit if and only if 1 and $1 + 2m$ are units in \mathbb{Z} . Therefore there are no cases where the class is a unit, q is odd and k is formally real.

Suppose k is not formally real, and q is odd. Then by the proposition $1 + m + me_{n,q}$ is a unit if and only if $1 + m - (-1)^n m$ is a unit, whereupon it is necessary and sufficient for n to be even. \square

5. THE HILTON-MILNOR SPLITTING

The James construction on a pointed simplicial set was introduced by I. James in [Jam55]. The idea of applying it in \mathbb{A}^1 homotopy theory, and thereby obtaining a weak equivalence $J(X) \simeq \Omega\Sigma X$ as in Proposition 5.2, is not original to us. We learned of it from A. Asok and J. Fasel, who attribute it to F. Morel.

Our presentation is based on that of [Whi12, Chapter VII.2]. An injection $\alpha : (1, 2, \dots, n) \rightarrow (1, 2, \dots, m)$ induces a map $\alpha_* : X^n \rightarrow X^m$. Let \sim denote the equivalence relation on $\coprod_{n=0}^{\infty} X^n$ generated by $x \sim \alpha_*(x)$ for all injections α . The James construction on X is $J(X) = \coprod_{n=0}^{\infty} X^n / \sim$. The construction $J(X)$ is the free monoid on the pointed simplicial set X . The k -simplices $J(X)_k$ of $J(X)$ are the free monoids on the pointed sets X_k , that is $J(X)_k = \coprod_{n=0}^{\infty} X_k^n / \sim$. The James construction is filtered by pointed simplicial sets $J_n(X)$, defined $J_n(X) = \coprod_{m=0}^n X^m / \sim$.

Define spaces $D_n(X)$ as the cofibers of sequences

$$J_{n-1}(X) \rightarrow J_n(X) \rightarrow D_n(X).$$

There are canonical weak equivalences $D_n(X) \rightarrow X^{\wedge n}$. Define $D(X) = \bigvee_{n=0}^{\infty} D_n(X)$.

Definition 5.1. For a pointed simplicial pre-sheaf X , define $J(X), J_n(X), D_n(X), D(X) \in \mathbf{sPre}$ by

$$J(X)(U) = J(X(U)), \quad J_n(X)(U) = J_n(X(U)), \quad D_n(X)(U) = D_n(X(U)), \quad D(X)(U) = D(X(U)).$$

Let $\ell : X = J_1(X) \rightarrow J(X)$ denote the map induced by the canonical maps $X(U) \rightarrow J(X(U))$ for $U \in \mathbf{Sm}$.

The *James construction* is then defined to be $L_{\mathbb{A}^1} J(X)$.

We learned the following result from A. Asok and J. Fasel.

Proposition 5.2. *Suppose X is a connected object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$. There is a natural isomorphism $J(X) \rightarrow \Omega\Sigma X$ in $\mathbf{ho}_{\mathbf{Nis}} \mathbf{sPre}(\mathbf{Sm}_k)_*$.*

Proof. For any object U in \mathbf{Sm}_k , there is a functorial weak equivalence $j : J(X(U)) \rightarrow F[X(U)]$ where $F[X(U)]$ is Milnor's construction, as laid out in [Wu10, Section 3.2]. There is then a functorial zig-zag of weak equivalences

$$F[X(U)] \rightarrow \Sigma X(U)_{\eta} \times_{\Sigma X} P\Sigma X \leftarrow \Omega\Sigma X(U)$$

as in [GJ99, Chapter V, Corollary 5.1]. Since these constructions are functorial, they induce a zig-zag of global weak equivalences, and therefore of local weak equivalences, from the presheaf $J(X)$ to $\Omega\Sigma X$. \square

Note that this natural isomorphism induces a natural isomorphism $J(X) \rightarrow \Omega\Sigma X$ in $\mathbf{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)_*$. It also follows immediately from this result that if $X \rightarrow Y$ is a local weak equivalence, then the functorial map $J(X) \rightarrow J(Y)$ is a local weak equivalence.

Corollary 5.3. *Suppose X is a connected object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$. Then there is a natural isomorphism*

$$\pi_i^{\mathbb{A}^1} J(X) \cong \pi_{i+1}^{\mathbb{A}^1} \Sigma X.$$

Proof. By Proposition 5.2, there is a natural isomorphism $\pi_i^{\mathbb{A}^1} J(X) \cong \pi_i^{\mathbb{A}^1} \Omega\Sigma X$. By definition, $\pi_i^{\mathbb{A}^1} \Omega\Sigma X = \pi_i L_{\mathbb{A}^1} \Omega\Sigma X$. Since ΣX is simplicially simply connected, $\Omega\Sigma X$ is simplicially connected. By Morel's connectivity theorem [Mor12, Theorem 6.38], $L_{\mathbb{A}^1} \Omega\Sigma X$ is also simplicially connected. Thus $\pi_0 L_{\mathbb{A}^1} \Omega\Sigma X \cong *$ is strongly \mathbb{A}^1 -invariant. By [Mor12, Theorem 6.46], it follows that the canonical morphism $L_{\mathbb{A}^1} \Omega\Sigma X \rightarrow \Omega L_{\mathbb{A}^1} \Sigma X$ is a simplicial weak equivalence. Thus there is a natural isomorphism $\pi_i L_{\mathbb{A}^1} \Omega\Sigma X \cong \pi_{i+1} L_{\mathbb{A}^1} \Sigma X$. Combining with the previous gives the claimed natural isomorphism $\pi_i^{\mathbb{A}^1} J(X) \cong \pi_{i+1}^{\mathbb{A}^1} \Sigma X$. \square

Corollary 5.4. *Suppose $X \rightarrow Y$ is an \mathbb{A}^1 weak equivalence of connected objects of $\mathbf{sPre}(\mathbf{Sm}_k)_*$, then the functorial map $J(X) \rightarrow J(Y)$ is an \mathbb{A}^1 weak equivalence.*

Proof. We will show that if $X \rightarrow Y$ is an \mathbb{A}^1 weak equivalence of connected objects, then $\Omega\Sigma X \rightarrow \Omega\Sigma Y$ is an \mathbb{A}^1 weak equivalence. Since there is a natural isomorphism $\Omega\Sigma X \cong J(X)$ in $\mathbf{ho\,sPre}(\mathbf{Sm}_k)_*$, and therefore in $\mathbf{ho_{\mathbb{A}^1}\,sPre}(\mathbf{Sm}_k)_*$, it will follow that $J(X) \rightarrow J(Y)$ is an isomorphism in $\mathbf{ho_{\mathbb{A}^1}\,sPre}(\mathbf{Sm}_k)_*$, as claimed.

The following is a sequence of local weak equivalences.

$$\begin{aligned} L_{\mathbb{A}^1}\Sigma X &\xrightarrow{\simeq} L_{\mathbb{A}^1}\Sigma Y && \text{(since } \mathbb{A}^1 \text{ localization is simplicial)} \\ \Omega L_{\mathbb{A}^1}\Sigma X &\xrightarrow{\simeq} \Omega L_{\mathbb{A}^1}\Sigma Y \\ L_{\mathbb{A}^1}\Omega\Sigma X &\xrightarrow{\simeq} L_{\mathbb{A}^1}\Omega\Sigma Y && \text{by [Mor12, Theorem 6.46]} \end{aligned}$$

but this is precisely what was to be shown. \square

Given $W, X \in \mathbf{sSet}_*$ and a map $f : (J_n W, J_{n-1} W) \rightarrow X$, we define the *combinatorial extension* of f

$$h(f) : J(W) \rightarrow J(X)$$

by following the procedure of [Jam55, 1.4, §2] (cf. [Whi12, Chapter VII.2]).

We first define the restriction of $h(f)$ to $J_m(W)$. For $m < n$, the restriction of $h(f)$ is the constant map. Suppose $m \geq n$. To an injection $(1, 2, \dots, n) \rightarrow (1, 2, \dots, m)$, we may associate a map $W^m \rightarrow W^n$ and therefore a map $W^m \rightarrow W^n \rightarrow J_n(W)$.

Consider the set of all $\binom{m}{n}$ increasing, n -term subsequences of $(1, 2, \dots, m)$. Order these by lexicographic ordering, reading from the right. Each sequence is an injective map $\{1, \dots, n\} \rightarrow \{1, \dots, m\}$. Taking the ordered product over all injections, we obtain a total map

$$W^m \rightarrow J_n(W)^{\binom{m}{n}}.$$

The $\binom{m}{n}$ -fold product of the map f is map

$$J_n(W)^{\binom{m}{n}} \rightarrow X^{\binom{m}{n}}.$$

We set the restriction of $h(f)$ to $J_m(W)$ to be

$$W^m \rightarrow J_n(W)^{\binom{m}{n}} \rightarrow X^{\binom{m}{n}} \rightarrow J_{\binom{m}{n}}(X) \rightarrow J(X).$$

One checks that this is well-defined.

This definition is functorial, and extends immediately to presheaves:

Definition 5.5. Given $W, X \in \mathbf{sPre}_*$ and a map $f : (J_n(W), J_{n-1}(W)) \rightarrow X$, we may define the *combinatorial extension* of f :

$$h(f) : J(W) \rightarrow J(X), \quad h(f)(U) = h(f(Y)).$$

For $X \in \mathbf{sSet}_*$, the cofiber sequences $J_{n-1}(X) \rightarrow J_n(X) \rightarrow D_n(X)$ induce natural maps

$$(J_n(X), J_{n-1}(X)) \rightarrow D_n(X).$$

For $X \in \mathbf{sPre}$, we thereby obtain maps $(J_n(X), J_{n-1}(X)) \rightarrow D_n(X)$, and consequently maps

$$j_n : J(X) \rightarrow J(D_n(X))$$

by combinatorial extension.

Let $i_n : J(D_n(X)) \rightarrow J(D(X))$ be the map induced by the canonical inclusions $D_n(X(U)) \rightarrow D(X(U))$. The monoid structure on $J(D(X(U)))$ induces multiplication maps $\mu_n : J(D(X))^n \rightarrow J(D(X))$. Consider the maps

$$\mu_{n+1} \prod_{m=0}^n i_m j_m : J(X) \rightarrow J(D(X))$$

for $n = 0, 1, 2, \dots$. It is important here that the product be ordered, and we declare it to be ordered by increasing values of m .

The composition of $i_n j_n$ with $J_{n-1}(X) \rightarrow J(X)$ is the constant based map. We'll say that the restriction of $i_n j_n$ to $J_{n-1}(X)$ is the constant map. It follows that $\mu_{n+1} \prod_{m=0}^n i_m j_m$ restricted to $J_{n-1}(X)$ is equal to the restriction of $\mu_{N+1} \prod_{m=0}^N i_m j_m$ to $J_{n-1}(X)$ for all $N \geq n$. Note that $J(X) = \text{colim } J_n(X)$. Thus we may define $f : J(X) \rightarrow J(D(X))$ by

$$f = \text{colim}_n \mu_{n+1} \prod_{m=0}^n i_m j_m.$$

For convenience, extend f to $f_+ : J(X)_+ \rightarrow J(D(X))$ by mapping the disjoint point via

$$* = X(U)^{\wedge 0} \rightarrow DX \rightarrow J(D(X)).$$

Taking the simplicial suspension of f_+ , we obtain

$$\Sigma f_+ : \Sigma(J(X)_+) \rightarrow \Sigma J(D(X)).$$

Let $|\cdot| : \mathbf{sSet} \rightarrow \mathbf{K}$ denote the geometric realization functor from simplicial sets to Kelly spaces, and let $\text{Simp} : \mathbf{K} \rightarrow \mathbf{sSet}$ be the right adjoint functor, which is the functor of singular simplices.

We claim that for any simplicial presheaf Y , for example $Y = D(X)$, there is an evaluation map

$$(9) \quad \Sigma J(Y) \rightarrow \text{Simp } |\Sigma Y|.$$

To see this, let $\Omega^M : \mathbf{Top} \rightarrow \mathbf{Top}$ denote the Moore loops functor, [CM95]. There is a strictly associative multiplication $\Omega^M \times \Omega^M \rightarrow \Omega^M$. Since taking Simp commutes with finite products, there is a strictly associative multiplication on $\text{Simp } \Omega^M$, and therefore an induced commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & \text{Simp } \Omega^M |\Sigma Y| \\ & \searrow & \nearrow \\ & J(Y) & \end{array}$$

Applying Σ , we obtain a map $\Sigma J(Y) \rightarrow \Sigma \text{Simp } \Omega^M |\Sigma Y|$. There is a natural transformation of functors $\Sigma \text{Simp} \rightarrow \text{Simp } \Sigma$ and so we have a composite

$$(10) \quad \Sigma J(Y) \rightarrow \Sigma \text{Simp } \Omega^M |\Sigma Y| \rightarrow \text{Simp } \Sigma \Omega^M |\Sigma Y|.$$

The counit of the adjunction between loops and suspension produces a natural transformation $\Sigma \Omega^M \rightarrow \text{id}$. Composing with (10) produces a map

$$\Sigma J(Y) \rightarrow \Sigma \text{Simp } \Omega^M |\Sigma Y| \rightarrow \text{Simp } \Sigma \Omega^M |\Sigma Y| \rightarrow \text{Simp } |\Sigma Y|,$$

which is what we claimed in (9).

Composing Σf_+ with (9) for $Y = D(X)$ produces a map

$$(11) \quad \Sigma J(X)_+ \rightarrow \text{Simp } |\Sigma D(X)|.$$

For each U , this map (11) evaluated at U is a weak equivalence; this is due to James, [Jam55, Theorem 5.6], and is presented in more recent terminology in [Whi12, VII Theorem 2.6]. Thus (11) is a weak equivalence

in the simplicial model structures on $\mathbf{sPre}(\mathbf{Sm}_k)_*$. Combining with the injective weak equivalence

$$\Sigma D(X) \rightarrow \text{Simp } |\Sigma D(X)|,$$

we have the zig-zag of injective weak equivalences

$$(12) \quad \Sigma J(X)_+ \longrightarrow \text{Simp } |\Sigma D(X)| \longleftarrow \Sigma D(X) .$$

We have shown:

Proposition 5.6. *Suppose X is a connected object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$. There is a canonical isomorphism*

$$\Sigma J(X)_+ \rightarrow \Sigma D(X)$$

in $\text{ho } \mathbf{sPre}(\mathbf{Sm}_k)_$.*

Corollary 5.7. *There is a canonical isomorphism $\sigma : J(X)_+ \rightarrow D(X)$ in $\text{ho}_{\text{Nis}} \mathbf{Spt}(\mathbf{Sm}_k)$.*

Remark 5.8. Here and subsequently we write $J(X)_+$, $D(X)$ in place of the stable $\Sigma^\infty J(X)_+$, $\Sigma^\infty D(X)$ whenever the context demands S^1 stable objects.

6. THE STABLE ISOMORPHISM

6.1. The diagonal. Let $\sigma : J(X)_+ \rightarrow D(X)$ denote the stable isomorphism in $\text{ho } \mathbf{Spt}(\mathbf{Sm}_k)$ of corollary 5.7. The category $\text{ho}_{\text{Nis}} \mathbf{Spt}(\mathbf{Sm}_k)$ is equipped with localization functors to $\text{ho}_{\mathbb{A}^1} \mathbf{Spt}(\mathbf{Sm}_k)$, $\text{ho}_p \mathbf{Spt}(\mathbf{Sm}_k)$ and $\text{ho}_{p, \mathbb{A}^1} \mathbf{Spt}(\mathbf{Sm}_k)$, this being the upshot of Section 3. We will denote the images of objects and morphisms under the various localization functors by the same notation as we use in the category $\text{ho}_{\text{Nis}} \mathbf{Spt}(\mathbf{Sm}_k)$, and in order to avoid confusion we will specify the category in which we are working.

Let $\Delta^q : J(X) \rightarrow J(X)^q$ denote the order- r diagonal of $J(X)$.

Definition 6.1. Let $\Delta_{i, (a_1, a_2, \dots, a_q)}^q(X)$ denote the composition in $\text{ho}_{\text{Nis}} \mathbf{Spt}(\mathbf{Sm}_k)$:

$$D_i X \longrightarrow D(X) \xrightarrow{\sigma^{-1}} J(X)_+ \xrightarrow{\Delta^q} (J(X)^q)_+ \xrightarrow{\bigwedge^q \sigma} \bigwedge^q D(X) \longrightarrow D_{a_1}(X) \wedge D_{a_2}(X) \wedge \dots \wedge D_{a_q}(X) .$$

Proposition 6.2 (Kuhn [Kuh01]). *If X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$ equipped with a co-H-structure, then*

$$\Delta_{i, (a_1, \dots, a_q)}^q(X) \cong *$$

in $\text{ho}_{\text{Nis}} \mathbf{Spt}(\mathbf{Sm}_k)$, unless $i = \sum_{j=1}^q a_j$.

Proof. This is part of Theorem 1.2 of [Kuh01]. □

For a positive number a and a_1, a_2, \dots, a_m , let $(a_1, a_2, \dots, a_m)^a$ denote the set of functions

$$\sigma : \{1, 2, \dots, a\} \rightarrow \{1, 2, \dots, m\}$$

such that $\sigma^{-1}(i)$ has cardinality a_i . note that $(a_1, a_2, \dots, a_m)^a$ is non-empty if and only if $a = \sum a_i$.

Given an element $\sigma \in (a_1, a_2, \dots, a_m)^a$, and a natural number $i \leq m$, write $\sigma^{-1}(i)$ as

$$\{\sigma^{-1}(i)_1, \sigma^{-1}(i)_2, \dots, \sigma^{-1}(i)_{a_i}\}$$

in such a way that $\sigma^{-1}(i)_j < \sigma^{-1}(i)_{j+1}$ for all $j \leq a_i - 1$. then define $\tilde{\sigma}$ to be the permutation on a letters sending $(1, 2, \dots, a)$ to $(\sigma^{-1}(1)_1, \sigma^{-1}(1)_2, \dots, \sigma^{-1}(1)_{a_1}, \sigma^{-1}(2)_1, \dots, \sigma^{-1}(2)_{a_2}, \dots, \sigma^{-1}(m)_{a_m})$.

For instance, if σ is the element of $(1, 2, 2)^5$ given by sending $2 \mapsto 1$, $1, 5 \mapsto 2$ and $3, 4 \mapsto 3$, then $\tilde{\sigma}$ is the permutation taking $(1, 2, 3, 4, 5)$ to $(2, 1, 5, 3, 4)$.

Suppose X is an objects of $\mathbf{sPre}(\mathbf{Sm}_k)_*$. For $\sigma \in \binom{a}{a_1, a_2, \dots, a_m}$, then define $e(\sigma) : X^{\wedge a} \rightarrow X^{\wedge a}$ to be the map induced by $\tilde{\sigma}$, and define $\text{sign}(\sigma)$ to be the number of pairs $r < k$ in $\{1, 2, \dots, a\}$ such that $\tilde{\sigma}(r) > \tilde{\sigma}(k)$. In the example given, $\text{sign}(\sigma)$ is the cardinality of $\{(1, 2), (3, 4), (3, 5)\}$, i.e. 3.

Proposition 6.3. *Suppose X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$. Let i, a_1, a_2, \dots, a_w be non-negative integers such that $i = \sum a_k$. Then*

$$\Delta_{i, (a_1, a_2, \dots, a_w)}^w(X) = \sum_{\sigma \in \binom{i}{a_1, a_2, \dots, a_w}} e(\sigma)$$

in $\text{ho}_{\text{Nis}} \mathbf{Spt}(\mathbf{Sm}_k)$.

Proof. This is Theorem 2.4 of [Kuh01] along with the observation that the result there is functorial, and may therefore be applied to the simplicial sets $X(\mathbf{U})$ as \mathbf{U} ranges over \mathbf{Sm}_k . \square

In the case where $X = S^{n+q\alpha}$, Remark 4.5 says that $e(\sigma)$ is $e_{n,q}^{\text{sign } \sigma}$. Proposition 6.3 then gives:

Corollary 6.4. *Suppose $X = S^{n+q\alpha}$. Let i, a_1, a_2, \dots, a_w be non-negative integers such that $i = \sum a_k$. Then*

$$\Delta_{i, (a_1, a_2, \dots, a_w)}^w(X) = \sum_{\sigma \in \binom{i}{a_1, a_2, \dots, a_w}} e_{n,q}^{\text{sign } \sigma}$$

in $\text{ho}_{\text{Nis}} \mathbf{Spt}(\mathbf{Sm}_k)$.

6.2. Combinatorics. We will have occasion to use an involution

$$\gamma : \binom{a_1 + a_2 + \dots + a_m}{a_1, a_2, \dots, a_m} \rightarrow \binom{a_1 + a_2 + \dots + a_m}{a_1, a_2, \dots, a_m}$$

which is defined as follows:

Take $\sigma \in \binom{a_1 + a_2 + \dots + a_m}{a_1, a_2, \dots, a_m}$, this is a function $\sigma : \{1, \dots, a_1 + a_2 + \dots + a_m\} \rightarrow \{1, \dots, m\}$. There are two possibilities

- (1) $\sigma(2i-1) = \sigma(2i)$ for all applicable i . In this case, we say $\gamma(\sigma) = \sigma$, so σ is fixed under the involution. We write $F_\gamma(a_1 + a_2 + \dots + a_m; a_1, a_2, \dots, a_m)$ for the set of fixed points, or F_γ when the coefficients are clear from the context.
- (2) Otherwise, there exists a least integer i such that $\sigma(2i-1) \neq \sigma(2i)$. We then let $\gamma(\sigma)$ be the function that agrees with σ except that $\gamma(\sigma)(2i-1) = \sigma(2i)$ and $\gamma(\sigma)(2i) = \sigma(2i-1)$.

If σ is not a fixed point of γ , then $\text{sign}(\sigma) + \text{sign}(\gamma(\sigma)) \equiv 1 \pmod{2}$, so that the number of elements in $\binom{a_1 + a_2 + \dots + a_m}{a_1, a_2, \dots, a_m}$ of even sign is given by the formula

$$\begin{aligned} \text{Number of elements of even sign} &= \frac{1}{2} \left(\left| \binom{a_1 + a_2 + \dots + a_m}{a_1, a_2, \dots, a_m} \right| - |F_\gamma| \right) + |F_\gamma| = \\ (13) \quad &= \frac{1}{2} \left(\left| \binom{a_1 + a_2 + \dots + a_m}{a_1, a_2, \dots, a_m} \right| + |F_\gamma| \right). \end{aligned}$$

We can often find ways to calculate $\left| \binom{a_1 + a_2 + \dots + a_m}{a_1, a_2, \dots, a_m} \right|$ and $|F_\gamma|$.

The cardinality of $\binom{x+y}{x, y}$ is the binomial coefficient $\frac{(x+y)!}{x!y!}$.

Proposition 6.5. *Among the elements of $\binom{x+y}{x, y}$, the number having even sign is*

$$\frac{1}{2} \left(\left| \binom{x+y}{x, y} \right| + \left| \binom{\lfloor \frac{x+y}{2} \rfloor}{\lfloor \frac{x}{2} \rfloor, \lfloor \frac{y}{2} \rfloor} \right| \right).$$

By virtue of our definitions, the second summand is 0 in the case where x and y are both odd.

Proof. We rely on the involution γ and (13). Since $|\binom{x+y}{x,y}|$ is known, it remains to calculate $|F_\gamma|$.

There are several cases to consider:

- (1) If x and y are both odd, then every number in $\{1, \dots, x+y\}$ forms part of a pair $(2i-1, 2i)$, and there must be at least one pair for which $\sigma(2i-1) \neq \sigma(2i)$, since $\sigma^{-1}(1)$ is odd. There are therefore no fixed points of the involution.
- (2) If x and y are both even, then every number in $\{1, \dots, x+y\}$ forms part of a pair $(2i-1, 2i)$. In order for σ to be fixed by γ , it must be the case that $\sigma(2i-1) = \sigma(2i)$ for all i . Defining $\tau \in \binom{(x+y)/2}{x/2, y/2}$ by the formula $\tau(i) = \sigma(2i)$, we see that there is a bijection between F_γ and $\binom{(x+y)/2}{x/2, y/2}$.
- (3) If one of x and y is even and the other odd—say for specificity that x is even and y odd—then σ is fixed under γ if $\sigma(x+y) = 2$ and $\sigma(2i-1) = \sigma(2i)$ for all $i \leq \lfloor \frac{x+y}{2} \rfloor$. Similarly to the previous case, there is a bijection in this case between F_γ and $\binom{(x+y-1)/2}{x/2, (y-1)/2}$.

Since $|F_\gamma|$ agrees in all cases with

$$\left| \binom{\lfloor \frac{x+y}{2} \rfloor}{\lfloor \frac{x}{2} \rfloor, \lfloor \frac{y}{2} \rfloor} \right|$$

the proposition is proved. \square

Proposition 6.6. *Let $X \simeq S^{n+q\alpha}$ be a motivic sphere, and suppose m, r are nonnegative integers with $0 \leq r < 1$. The diagonal map $\Delta_{2m+r, (2m, r)}^2 : X^{\wedge p m + r} \rightarrow X^{\wedge p m} \wedge X^{\wedge r}$ is an isomorphism in $\mathrm{ho}_{2, \mathbb{A}^1} \mathbf{Spt}(\mathbf{Sm}_k)$.*

Proof. We have calculated $\Delta_{2m+r, (2m, r)}^2$ in Corollary 6.4 and Proposition 6.5. If $r = 0$, we find $\Delta_{2m, (2m, 0)}^2 = 1$ in $\mathrm{GW}(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}$.

When $r = 1$, we find $\Delta_{2m+1, (2m, 1)}^2 = (m+1) + me_{n,q}$. This is a unit in $\mathrm{GW}(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}$ by Corollary 4.7. \square

The same calculations, referring to Corollary 4.8, show the following:

Proposition 6.7. *Let $X \simeq S^{n+q\alpha}$ be a motivic sphere. Assume one of the following two conditions holds:*

- (1) n is odd and q is even,
- (2) $n+q$ is odd and the ground field k is not formally real.

Suppose m, r are nonnegative integers with $0 \leq r < 1$. The diagonal map $\Delta_{2m+r, (2m, r)}^2 : X^{\wedge 2m+r} \rightarrow X^{\wedge 2m} \wedge X^{\wedge r}$ is an isomorphism in $\mathrm{ho}_{\mathbb{A}^1} \mathbf{Spt}(\mathbf{Sm}_k)$.

Definition 6.8. We impose a total order on the elements of $\binom{a_1 + \dots + a_m}{a_1, a_2, \dots, a_m}$ by declaring $\sigma < \sigma'$ if $\sigma(j) = \sigma'(j)$ for all $j \leq k$ and $\sigma(k) < \sigma'(k)$.

Following [Kuh87], define a regular (r, s) -set of size m to be a set, $\{S_1, \dots, S_s\}$, of subsets of $\{1, \dots, m\}$ satisfying

- (1) $|S_i| = r$ for all i
- (2) $\bigcup_{i=1}^s S_i = \{1, \dots, m\}$.

Let $L(r, s, m)$ denote the set of all regular (r, s) sets of size m . This goes by the name $B(B(m, s), r)$ in [Kuh87], and the discussion that follows here is a much reduced version of the discussion to be found there. In particular, we concentrate on the case where $r = 2$ and $m = 2s$.

There is a cover $\phi : \binom{2s}{2, \dots, 2} \rightarrow L(2, s, 2s)$, the source being the set of ordered partitions of $\{1, \dots, 2s\}$ into s disjoint subsets of cardinality 2, and the latter be the set of unordered partitions. There is an S_s -action on functions $\sigma : \{1, \dots, 2s\} \rightarrow \{1, \dots, s\}$ induced from the action on the target; and the orbits of this action are in bijective correspondence with $L(2, s, 2s)$. For any $\lambda \in L(2, s, 2s)$, define $\text{sign}(\lambda)$ to be $\text{sign}(\sigma)$ where σ is the least element, in the order of Definition 6.8, of $\binom{2s}{2, \dots, 2}$ that maps to λ under ϕ .

Write $E(2, s)$ and $O(2, s)$ for the number of elements in $L(2, s, 2s)$ having even and odd sign, respectively. Trivially, $E(2, 1) = 1$ and $O(2, 1) = 0$.

Proposition 6.9. *The quantities $E(2, s)$ and $O(2, s)$ satisfy $E(2, s) = O(2, s) + 1$.*

Proof. An element $\lambda \in E(2, s)$ is a partition of $\{1, \dots, 2s\}$ into s disjoint subsets $\{S_1, \dots, S_s\}$. One orders these subsets in ascending order of their least members. The quantity $\text{sign}(\lambda)$ is the number of pairs of numbers $j_1 < j_2$ such that $j_1 \in S_{\ell_1}$ and $j_2 \in S_{\ell_2}$ with $S_{\ell_1} > S_{\ell_2}$. We can set up an involution $\bar{\gamma}$ on $L(2, s, 2s)$ by observing that γ descends from $\binom{2s}{2, \dots, 2}$.

Explicitly, if $\lambda = \{S_1, \dots, S_s\} \in L(2, s, 2s)$ is not the partition $\lambda_0 = \{\{1, 2\}, \{3, 4\}, \dots, \{2s-1, 2s\}\}$ then there is a least pair of integers $(2i-1, 2i)$ such that $2i-1$ and $2i$ lie in different sets S_i, S_j with $i \neq j$. Let $\bar{\gamma}(\lambda)$ be the partition obtained from λ by interchanging $2i-1$ and $2i$. The exceptional partition, λ_0 , is the unique fixed point of $\bar{\gamma}$.

Observe that if $\lambda \neq \lambda_0$, then $\text{sign}(\lambda) + \text{sign}(\bar{\gamma}(\lambda)) \equiv 1 \pmod{2}$. Since $\text{sign}(\lambda_0) = 0$ is even, it follows that $E(2, s) - 1 = O(2, s)$, as asserted. \square

Remark 6.10. The cardinality of $\binom{2s}{2, \dots, 2}$ is $\frac{(2s)!}{2^s}$, and that of $L(2, s, 2s)$ is $\frac{(2s)!}{s!2^s}$. Explicitly therefore

$$E(2, s) = \frac{1}{2} \left(\frac{(2s)!}{s!2^s} - 1 \right) + 1 = \frac{(2s)!}{s!2^{s+1}} + \frac{1}{2}$$

and

$$O(2, s) = \frac{1}{2} \left(\frac{(2s)!}{s!2^s} - 1 \right) = \frac{(2s)!}{s!2^{s+1}} - \frac{1}{2}.$$

The quantity $E(2, s) + O(2, s) = \frac{(2s)!}{s!2^s}$ is the product of the first s odd integers: $(2s-1)(2s-3) \dots (5)(3)(1)$. The fact that this is a unit in $\mathbb{Z}_{(2)}$ appears in the classical study of j_2 .

6.3. Decomposing the second James–Hopf map.

Definition 6.11. Let $\alpha_{i,s}^2 : D_i(X) \rightarrow D_s(X^{\wedge 2})$ denote the composition in $\text{ho}_{\text{Nis}} \mathbf{Spt}(\mathbf{Sm}_k)$

$$D_i X \longrightarrow D(X) \xrightarrow{\sigma^{-1}} J(X)_+ \xrightarrow{j_2} J(X^{\wedge 2})_+ \xrightarrow{\sigma} D(X^{\wedge 2}) \longrightarrow D_s(X^{\wedge 2}).$$

For example, we have

$$(14) \quad \alpha_{2,1}^2 = \text{id}_{D_2(X)}$$

by the commutative diagram

$$\begin{array}{ccc} J(X) & \xrightarrow{j_2} & J(X^{\wedge 2}) \\ \uparrow & & \uparrow \\ J_2(X) & \longrightarrow & J(X^{\wedge 2}) \end{array}$$

where the lower horizontal map is the composite

$$J_2(X) \rightarrow J_2(X)/J_1(X) = D_2(X) \cong X^{\wedge 2} \xrightarrow{\text{id}} X^{\wedge 2} \cong D_1(X^{\wedge 2}).$$

The stable weak equivalence σ induces a stable weak equivalence

$$\bigwedge^r \sigma : (J(X)^q)_+ \rightarrow \bigwedge^q D(X) \simeq \bigvee_{\alpha_1, \alpha_2, \dots, \alpha_q} (D_{\alpha_1}(X) \wedge D_{\alpha_2}(X) \wedge \dots \wedge D_{\alpha_q}(X)).$$

Proposition 6.12. *Let $X \simeq S^{n+q\alpha}$ be a motivic sphere with $n \geq 1$. Let $r \geq 2$ be an integer. The maps in $\mathrm{ho}_{\mathrm{Nis}} \mathbf{Spt}(\mathbf{Sm}_k)$*

$$D_i(X) \xrightarrow{\quad} J(X) \xrightarrow{j_2} J(X^{\wedge 2}) \xrightarrow{\quad} D_s(X^{\wedge 2})$$

$\xrightarrow{\quad a_{r,q}^2 \quad}$

of agree in $\mathrm{ho}_{\mathbb{A}^1} \mathbf{Spt}(\mathbf{Sm}_k)$ with

$$a_{i,s}^2 = \begin{cases} E(2, s) + O(2, s)e_{n,q} & \text{if } i = 2s, \\ * & \text{otherwise.} \end{cases}$$

Proof. The case where $i \neq 2s$ follows from Corollary 6.3(1) and (3) of [Kuh87].

When $i = 2s$, then by Theorem 6.2 of [Kuh87], the class $a_{i,s}^2$ is equal to the sum of the classes of permutations of $X^{\wedge 2s}$ associated to regular $(2, s)$ sets of size $2s$. Of these, $E(2, s)$ are even permutations, and therefore equivalent to the identity, and $O(2, s)$ are odd, and therefore equivalent to the single interchange $e_{n,q}$. \square

Corollary 6.13. *The map $a_{2s,s}^2$ is an isomorphism in $\mathrm{ho}_{2,\mathbb{A}^1}(\mathbf{Spt}(\mathbf{Sm}_k))$.*

Proof. The map in question is $E(2, s) + O(2, s)e_{n,r}$. Since $E(2, s) = O(2, s) + 1$ by Proposition 6.9, it follows from Corollary 4.7 that it is a unit in $\mathrm{GW}(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}$. \square

Remark 6.14.

Similarly, we have the following corollary:

Corollary 6.15. *If $X = S^{n+q\alpha}$ in $\mathrm{GW}(k)$ is a motivic sphere and one of the following conditions is satisfied:*

- (1) n is even and q is odd,
- (2) $n + q$ is odd and the field k is not formally real

then $a_{2s,s}^2$ represents an isomorphism in $\mathrm{ho}_{\mathbb{A}^1}(\mathbf{Spt}(\mathbf{Sm}_k))$.

Proof. This follows from Proposition 6.9 and Corollary 4.8. \square

6.4. The Stable Weak Equivalence. Let X be of the form $S^{n+q\alpha}$ for $n \geq 1$, $q \geq 2$. Let $e_{n,r}$ be the class $(-1)^{n+q} \langle -1 \rangle^q$.

Fix a set of primes, P . All objects and maps in this section belong to the category $\mathrm{ho}_{P,\mathbb{A}^1} \mathbf{Spt}(\mathbf{Sm}_k)$, unless otherwise stated. If P is the set of all primes, then $\mathrm{ho}_{P,\mathbb{A}^1}(\mathbf{Spt}(\mathbf{Sm}_k)) = \mathrm{ho}_{\mathbb{A}^1}(\mathbf{Spt}(\mathbf{Sm}_k))$. This case and the case $P = \{(2)\}$ are the two cases that are applied in subsequent sections of this paper.

Write $b_+ : J(X)_+ \cong \bigvee_{i=0}^{\infty} X^{\wedge i} \rightarrow X_+$ for the projection map.

We can construct a stable map $c = (j_+ \times b_+) \circ \Delta$

$$J(X)_+ \xrightarrow{\quad \Delta \quad} (J(X) \times J(X))_+ \xrightarrow{\quad} (J(X) \times X)_+ \xrightarrow{\quad} (J(X^{\wedge 2}) \times X)_+.$$

$\xrightarrow{\quad c \quad}$

Here Δ is the image in $\mathrm{ho}_{2,\mathbb{A}^1} \mathbf{Spt}(\mathbf{Sm}_k)$ of the diagonal map $J(X)_+ \rightarrow (J(X) \times J(X))_+$ in $\mathbf{sPre}(\mathbf{Sm}_k)_*$, and j is the James–Hopf map $j : J(X) \rightarrow J(X^{\wedge 2})$ in $\mathbf{sPre}(\mathbf{Sm}_k)$. Since j_+ is a map in $\mathbf{sPre}(\mathbf{Sm}_k)_*$, we can form the product map $(J(X) \times X)_+ \rightarrow (J(X^{\wedge 2}) \times X)_+$ in $\mathrm{ho}_{\mathbb{A}^1} \mathbf{Spt}(\mathbf{Sm}_k)(\mathbf{Sm}_k)$ by means of the action of $\mathbf{sPre}(\mathbf{Sm}_k)_*$ on $\mathbf{Spt}(\mathbf{Sm}_k)$.

Both $J(X)_+$ and $(J(X^{\wedge 2}) \times X)_+$ are isomorphic in the homotopy category $\mathrm{ho}_{2,\mathbb{A}^1} \mathbf{Spt}(\mathbf{Sm}_k)$ to the spectrum $\bigvee_{i=0}^{\infty} X^{\wedge i}$. To see the latter, decompose

$$\begin{aligned} (J(X^{\wedge 2}) \times X)_+ &\cong S^0 \vee J(X^{\wedge 2}) \vee X \vee (J(X^{\wedge 2}) \wedge X) \cong \\ (15) \quad &\cong S^0 \vee \left(\bigvee_{i=1}^{\infty} X^{\wedge 2i} \right) \vee X \vee \left(\bigvee_{i=1}^{\infty} X^{\wedge 2i+1} \right). \end{aligned}$$

Fix standard isomorphisms $J(X)_+ \cong \bigvee_{i=0}^{\infty} X^{\wedge i} \cong (J(X^{\wedge 2}) \times X)_+$ in $\mathrm{ho}_{\mathbb{A}^1} \mathbf{Spt}(\mathbf{Sm}_k)$.

Proposition 6.16. *Fix a sphere $X = S^{n+q\alpha}$. If elements $m + 1 + me_{n,q}$, where m is an integer, are units in $\mathrm{GW}(k) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, then the map c is a weak equivalence.*

Proof. Consider the ring

$$R = \mathrm{End}_{\mathrm{ho}_{2,\mathbb{A}^1} \mathbf{Spt}(\mathbf{Sm}_k)} \left(\bigvee_{i=0}^{\infty} X^{\wedge i} \right).$$

We wish to show that $(j_+ \times b_+) \circ \Delta$ is a unit of this ring. We may write

$$R = \prod_{i=0}^{\infty} \mathrm{Hom}_{\mathrm{ho}_{2,\mathbb{A}^1} \mathbf{Spt}(\mathbf{Sm}_k)} \left(X^{\wedge i}, \bigvee_{j=0}^{\infty} X^{\wedge j} \right)$$

and

$$\bigvee_{j=0}^{\infty} X^{\wedge j} \simeq \bigvee_{j=0}^i X^{\wedge j} \vee \bigvee_{j=i+1}^{\infty} X^{\wedge j}.$$

It follows from the Hurewicz theorem that $[X^{\wedge i}, \bigvee_{j=i+1}^{\infty} X^{\wedge j}] = 0$, and so

$$\mathrm{Hom}_{\mathrm{ho}_{2,\mathbb{A}^1} \mathbf{Spt}(\mathbf{Sm}_k)} \left(X^{\wedge i}, \bigvee_{j=0}^{\infty} X^{\wedge j} \right) = \bigoplus_{j=0}^i \mathrm{Hom}_{\mathrm{ho}_{2,\mathbb{A}^1} \mathbf{Spt}(\mathbf{Sm}_k)} (X^{\wedge i}, X^{\wedge j}),$$

so that $R = \prod_{i=0}^{\infty} \bigoplus_{j=0}^i \pi_{in+iq\alpha}(S^{jn+jq\alpha})$. We may represent elements of R as infinite, upper-triangular matrices $(d_{i,j})$ such that $d_{i,j} \in \pi_{in+iq\alpha}(S^{jn+jq\alpha})$ by decreeing $d_{i,j} = 0$ whenever $i < j$. It follows from the usual algebra of matrix multiplication that an element of R is a unit if and only if the terms $d_{i,i} \in \pi_{in+iq\alpha}(S^{in+iq\alpha})$ are units for all i .

The invertibility of c in R may be deduced from the classes $d_{i,i}$ appearing in this diagram

$$(16) \quad \begin{array}{ccccccc} & & & c & & & \\ & & & \curvearrowright & & & \\ J(X)_+ & \xrightarrow{\Delta} & (J(X) \times J(X))_+ & \longrightarrow & (J(X) \times X)_+ & \longrightarrow & (J(X^{\wedge 2}) \times X)_+ \\ & \uparrow & & & & & \downarrow \\ X^{\wedge i} & \xrightarrow{\quad d_{i,i} \quad} & & & & & X^{\wedge i}, \end{array}$$

where the unmarked arrows are inclusion and projection maps.

We can factor $d_{i,i}$ in diagram (16) as

$$(17) \quad \begin{array}{ccccccc} & & & c & & & \\ & & & \curvearrowright & & & \\ J(X)_+ & \xrightarrow{\Delta} & (J(X) \times J(X))_+ & \xrightarrow{\quad} & (J(X) \times X)_+ & \xrightarrow{\quad} & (J(X^{\wedge 2}) \times X)_+ \\ & \uparrow & \uparrow & & & & \downarrow \\ X^{\wedge i} & \xrightarrow{f} & \bigvee_{n=0}^i X^{\wedge i-n} \wedge X^{\wedge n} & \xrightarrow{\quad} & X^{\wedge i}, & & \\ & & & d_{i,i} & & & \end{array}$$

where f is the wedge sum of maps $\Delta_{i,(i-n,n)}^2 : X^{\wedge i} \rightarrow X^{\wedge i-n} \wedge X^{\wedge n}$ as n varies. This factorization follows from Proposition 6.2.

We can further factorize $d_{i,i}$ because the map $\Sigma^\infty(J(X) \times J(X))_+ \rightarrow \Sigma^\infty(J(X) \times X)_+$ is identity on the first and projection on the second factor:

$$(18) \quad \begin{array}{ccccccc} & & & c & & & \\ & & & \curvearrowright & & & \\ J(X)_+ & \xrightarrow{\Delta} & (J(X) \times J(X))_+ & \xrightarrow{\quad} & (J(X) \times X)_+ & \xrightarrow{\quad} & (J(X^{\wedge 2}) \times X)_+ \\ & \uparrow & \uparrow & & \uparrow & & \downarrow \\ X^{\wedge i} & \xrightarrow{f} & \bigvee_{n=0}^i X^{\wedge i-n} \wedge X^{\wedge n} & \xrightarrow{\quad} & (X^{\wedge i} \wedge X^{\wedge 0}) \vee (X^{\wedge i-1} \wedge X) & \xrightarrow{\quad} & X^{\wedge i}, \\ & & & d_{i,i} & & & \end{array}$$

Write $i = 2m + s$ where $s \in \{0, 1\}$. By use of Proposition 6.12, we deduce that the bottom row can be further factored as

$$(19) \quad \begin{array}{ccccccc} & & & d_{i,i} & & & \\ & & & \curvearrowright & & & \\ X^{\wedge i} & \xrightarrow{f} & \bigvee_{n=0}^i X^{\wedge i-n} \wedge X^{\wedge n} & \xrightarrow{\quad} & (X^{\wedge i} \wedge X^{\wedge 0}) \vee (X^{\wedge i-1} \wedge X) & \xrightarrow{\quad} & X^{\wedge i} \\ & & & & \downarrow & \nearrow a_{2m,m}^2 \wedge \text{id} & \\ & & & & X^{\wedge 2m} \wedge X^{\wedge s} & & \end{array}$$

It follows that $d_{i,i}$ factors as $(a_{2m,m}^2 \wedge \text{id}) \circ \Delta_{i,(2m,s)}^2$, and since both these maps are isomorphisms by hypothesis, so too is $d_{i,i}$, and therefore so too is $c = (j_+ \times b_+) \circ \Delta$. \square

Remark 6.17. The hypothesis of the Proposition that elements of the form $(m+1) + me_{n,q} \in \text{GW}(k) \otimes_{\mathbb{Z}} \mathbb{Z}_P$ be units holds in particular in the following cases:

- (1) The ring \mathbb{Z}_P is $\mathbb{Z}_{(2)}$ or \mathbb{Q} . In this case, the hypothesis holds by Corollary 4.7.
- (2) The integer n is odd and the integer q is even. In this case, $e_{n,q} = -1$, and the hypothesis holds by Corollary 4.8.
- (3) The integer $n+q$ is odd, and the field k is not formally real. Again, the hypothesis holds in this case by Corollary 4.8.

Remark 6.18. If X is an object in $\mathbf{sPre}(\mathbf{Sm}_k)_*$, there is an action of the symmetric group S_n on $X^{\wedge n}$. In the case where X is a motivic sphere, this action factors through the sign representation of S_n . The fact that c is a 2-local weak equivalence depends on this fact, as we can see in the following example.

Example 6.19. Let X be the simplicial set $X = S^2 \vee S^2$. The map $S_n \rightarrow [X^{\wedge n}, X^{\wedge n}]$ is injective because the action of S^n on $H^{2n}(X^{\wedge n}, \mathbb{Q}) \cong H^2(X, \mathbb{Q})^{\otimes n} \cong \mathbb{Q}^{2^n}$ contains a direct sum of two copies of the permutation

representation of S_n over \mathbb{Q} as summands. These two copies can be described as follows. The wedge product $X^{\wedge n}$ is the direct sum of copies of S^{2n} indexed by n -tuples of elements of $\{1, 2\}$. The n -tuples which have a single 1 and the rest 2's form one of the summands, and the other is obtained by switching the roles of 1 and 2.

The objects $J(X)_+$ and $J(X^{\wedge 2})_+$ split stably as $\bigvee_{i=0}^{\infty} X^{\wedge i}$ and $\bigvee_{i=0}^{\infty} X^{\wedge 2i}$ respectively. The second James–Hopf map

$$j_2 : \bigvee_{i=0}^{\infty} X^{\wedge i} \rightarrow \bigvee_{i=0}^{\infty} X^{\wedge 2i}$$

restricts to a map $\alpha_{4,2}^2 : X^{\wedge 4} \rightarrow X^{\wedge 4}$. The paper [Kuh87] calculates this map explicitly as the sum of permutations $\alpha_{4,2}^2 = \sum_{\sigma \in \binom{4}{2,2}} e(\sigma)$. Note that $\binom{4}{2,2}$ is in bijection with $\{((1, 2)(3, 4)), ((1, 3)(2, 4)), ((1, 4)(2, 3))\}$ under the bijection sending $((a, b), (c, d))$ to the map sending a and b to 1 and sending c and d to 2. Using cycle notation for permutations, and representing the identity by e , this sum is

$$\alpha_{4,2}^2 = e + (23) + (243).$$

The induced map on the singular cohomology $H^8(X^{\wedge 4}, \mathbb{Q}) \cong \mathbb{Q}^{16}$ is not of full rank, since $e + (23) + (243)$ is not an isomorphism on the permutation representation, namely on either of the submodules mentioned above $\alpha_{4,2}^2$ acts by the matrix

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

which has determinant 0.

The map induced by j_2 on rational cohomology $H^8(J(X^{\wedge 2}), \mathbb{Q}) \rightarrow H^8(J(X), \mathbb{Q})$ is not an isomorphism in this case, and is in particular not injective, and so the analogue of Proposition 6.16 fails in this case, even \mathbb{Q} -locally.

$$\begin{array}{cccccccccc} H^3(\text{hofib}(j_2), \mathbb{Q}) \cong 0 & & & & & & & & & 0 \\ \mathbb{Q}^2 & \searrow^{d_4=0} & & & & & & & & \mathbb{Q}^8 \\ 0 & & & & & & & & & \\ \mathbb{Q} & & 0 & 0 & 0 & \rightarrow & \mathbb{Q}^4 & 0 & 0 & 0 & \mathbb{Q}^{16} \cong H^8(J(X^{\wedge 2}), \mathbb{Q}) \end{array}$$

FIGURE 1. The first four rows, nine columns of the E_2 -page of the Serre spectral sequence for $H^*(\cdot, \mathbb{Q})$ associated to $\text{hofib}(j_2) \rightarrow J(X) \xrightarrow{j_2} J(X^{\wedge 2})$.

Moreover, associated to the fiber sequence

$$\text{hofib}(j_2) \rightarrow J(X) \xrightarrow{j_2} J(X^{\wedge 2})$$

there is a Serre spectral sequence for rational cohomology, part of which is shown in Figure 1. We have shown that the edge map $H^8(J(X^{\wedge 2}), \mathbb{Q}) \rightarrow H^8(J(X), \mathbb{Q})$ is not injective. Since the edge map is not injective, it is not the case that the spectral sequence collapses at the E_2 page, and since $H^*(J(X^{\wedge 2}), \mathbb{Q})$ is concentrated in even degrees, it follows that $H^*(\text{hofib}(j_2), \mathbb{Q})$ is not also concentrated in even degrees. In particular, $\text{hofib}(j_2)$ does not have the same rational cohomology as $X = S^2 \wedge S^2$, showing that even the \mathbb{Q} -local version of the EHP sequence does not hold for a general space X .

7. FIBER OF THE JAMES-HOPF MAP

In Section 7.3, we will have two fiber sequences $F \rightarrow E \rightarrow Y$ and $X \rightarrow X \times Y \rightarrow Y$ with the same base Y and a stable weak equivalence between the total spaces E and $X \times Y$, which is compatible with the map to the base. We will show that in fact the fibers are stably weakly equivalent as well (Proposition 7.19). For this, it is natural to ask for a Serre spectral sequence, as the Serre spectral sequence gives a good way to measure the size of the total space of a fibration in terms of the size of the base and the fiber. Since the base spaces of the fibrations f and p are the same, and their total spaces are the same size, a Serre spectral sequence would give us a tool with which to attempt to ‘cancel off the base space’ and conclude that the fibers have the same size. The purpose of the first part of this section is to show that enough of these ideas remain available in \mathbb{A}^1 -homotopy theory. In Section 7.1, we construct a spectral sequence to substitute for the Serre spectral sequence. We develop needed properties in Section 7.2, and in Section 7.3, we show that the desired cancelation is possible.

7.1. A spectral sequence. Let α be a left Bousfield localization of the global model structure on $\mathbf{sPre}(\mathbf{Sm}_k)$. There is an associated stable model structure on the category of S^1 -spectra, $\mathbf{Spt}(\mathbf{Sm}_k)$. See Section 2.4. Let $\mathcal{H}_i : \mathbf{Spt}(\mathbf{Sm}_k) \rightarrow \mathbf{Sh}_{\text{Nis}}$ be an α -corepresentable homology theory, corepresented by a spectrum E , so that $\mathcal{H}_i(F)$ is the Nisnevich sheaf associated to the presheaf

$$U \mapsto [\Sigma^\infty S^i \wedge E \wedge \Sigma^\infty U_+, F]_{\alpha, s}.$$

We write $\mathcal{H}_i(X)$ for $\mathcal{H}_i(\Sigma^\infty X)$ when X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$.

Since left Bousfield localization does not change which maps are cofibrations, the notions of global cofibration, Nisnevich local cofibration, \mathbb{A}^1 -cofibration, and α -cofibration for \mathbf{sPre} are the same. For $X_1 \rightarrow X_2$ a cofibration with respect to these model structures, the cofiber C is the push-out $C = \text{colim } X_1 \longrightarrow X_2$. A

$$\downarrow$$

$$*$$

sequence is said to be a cofiber (respectively fiber) sequence up to homotopy, if the sequence is isomorphic in the homotopy category to a cofiber (respectively fiber) sequence.

Proposition 7.1. *The homology theory \mathcal{H}_* has the following properties:*

- (1) \mathcal{H}_i takes α weak equivalences to isomorphisms.
- (2) Given a cofibration $X_1 \rightarrow X_2$ with cofiber C in α , there is a natural long exact sequence of sheaves of abelian groups

$$\dots \rightarrow \mathcal{H}_i X_1 \rightarrow \mathcal{H}_i X_2 \rightarrow \mathcal{H}_i C \rightarrow \mathcal{H}_{i-1} X_1 \rightarrow \dots$$

- (3) As a special case of (2), we see that $\mathcal{H}_i(\Sigma X) \simeq \mathcal{H}_{i-1}(X)$ for $X \in \mathbf{sPre}(\mathbf{Sm}_k)_*$.

Assumption 7.2. We assume that \mathcal{H}_* satisfies two further axioms.

- (1) **Boundedness:** $\mathcal{H}_i(X) = 0$ for $i < 0$ for all objects X in $\mathbf{sPre}(\mathbf{Sm}_k)$.
- (2) **Compactness:** $\text{colim } \mathcal{H}_i(X_j) = \mathcal{H}_i(\text{colim } X_j)$ for all filtered diagrams $\{X_i\}$ in $\mathbf{sPre}(\mathbf{Sm}_k)$.

These axioms are satisfied by $\mathcal{H}_i = \pi_{i+j, \alpha}^{s, \mathbb{A}^1}$ and $\mathcal{H}_i = \pi_{i+j, \alpha}^{s, P, \mathbb{A}^1}$: we take α to be the \mathbb{A}^1 injective structure or the P - \mathbb{A}^1 injective structure respectively. In each case, $E = \Sigma^\infty \mathbb{G}_m^{\wedge j}$. The boundedness axiom follows from Proposition 2.19, the compactness from Proposition 2.17 or Proposition 3.26.

For $f : X \rightarrow Y$ a global fibration, we construct a spectral sequence $E_{ij}^r \Rightarrow \mathcal{H}_{i+j} X$. This spectral sequence for $\alpha = \mathbb{A}^1$ or its P -localizations will have the property that it relates α -homotopy invariant information about the total space with α -homotopy invariant information about the fiber and more delicate information about the base.

Let $\Delta_{\leq n}$ be the full subcategory of Δ on the objects $\{0, 1, \dots, n\}$. The n -skeleton $sk_n : \mathbf{sSet} \rightarrow \mathbf{sSet}$ can be defined as the composite of the n -truncation functor $\mathbf{sSet} \rightarrow \text{Fun}(\Delta_{\leq n}^{\text{op}}, \mathbf{Set})$ with its left adjoint. Given a simplicial presheaf X , define $sk_n X \in \mathbf{sPre}$ by $U \mapsto sk_n X(U)$. For notational simplicity, define $sk_n X$ for $n < 0$ by $sk_n X = \emptyset$.

Definition 7.3. Let $f : X \rightarrow Y$ be a fibration in the global model structure between pointed simplicial presheaves. We define the following spectral sequence

$$(E_{ij}^r, d^r : E_{i+r,j}^r \rightarrow E_{i,j+r-1}^r).$$

The cofibrations

$$sk_0 Y \rightarrow sk_1 Y \rightarrow \dots \rightarrow sk_n Y \rightarrow \dots Y$$

pull-back to cofibrations

$$sk_0 Y \times_Y X \rightarrow sk_1 Y \times_Y X \rightarrow \dots \rightarrow sk_n Y \times_Y X \rightarrow \dots Y \times_Y X = X.$$

The cofiber sequences

$$sk_{n-1} Y \times_Y X \rightarrow sk_n Y \times_Y X \rightarrow C_n$$

for $n \geq 0$ give rise to the long exact sequences of (2) Proposition 7.1, which form an exact couple

$$\begin{array}{ccc} \bigoplus_{i,n=0}^{\infty} \mathcal{H}_i(sk_n Y \times_Y X) & \xrightarrow{\quad} & \bigoplus_{i,n=0}^{\infty} \mathcal{H}_i(sk_n Y \times_Y X) \\ & \searrow \quad \swarrow & \\ & \bigoplus_{i,n=0}^{\infty} \mathcal{H}_i C_n & \end{array}$$

This exact couple gives rise to the spectral sequence $E_{i,j}^r$ with $E_{i,j}^1 = \mathcal{H}_{i+j} C_i$.

Remark 7.4. To be more explicit about the choice of base points in the definition of the spectral sequence just constructed, note that the cofiber sequences

$$sk_{n-1} Y \times_Y X \rightarrow sk_n Y \times_Y X \rightarrow C_n$$

are pointed for $n \geq 1$ as the chosen base points of X and Y are in the 0-skeleton, and induce a base point in C_n . By extending the definition of \mathcal{H}_i to take the value of 0 on \emptyset , we obtain an analogous long exact sequence for $n \geq 0$. These long exact sequences form the exact couple.

For $U \in \mathbf{Sm}_k$, let $L_n Y(U) \in \mathbf{sSet}$ denote the n th latching object, defined $L_n Y(U) = (sk_{n-1} Y(U))_n$, and let $N_n Y(U)$ be the set of non-degenerate n -simplices of $Y(U)$, defined $N_n Y(U) = Y_n(U) - L_n Y(U)$.

Let F_y denote the fiber of f over a point $y : * \rightarrow Y$ of Y . More generally if $y : U \rightarrow Y$ is a map, let $F_y = U \times_Y X$. Let Δ^n denote the constant simplicial presheaf on the standard n simplex in \mathbf{sSet} . For $y \in Y_n(U)$, we have an associated map $y : \Delta^n \times U \rightarrow Y$. Restricting y along the map $0 \times U \rightarrow \Delta^n \times U$, we obtain $F_{y(0)}$ by the previous, and thus $F_{y(0)}(U)$, which is computed by the pull-back diagram

$$\begin{array}{ccc} F_{y(0)}(U) & \longrightarrow & X(U) \\ \downarrow & & \downarrow f(U) \\ * & \xrightarrow{y(0)(U)} & Y(U). \end{array}$$

Despite the fact that $Y_n(-) - L_n Y(-)$ does not necessarily define a presheaf,

$$U \mapsto \bigvee_{y \in N_n Y(U)} (F_{y(0)}(U)_+ \wedge (\Delta^n / \partial \Delta^n))$$

is a presheaf because it could equally well be written

$$\frac{\bigvee_{y \in Y_n(\mathcal{U})} (F_{y(0)}(\mathcal{U})_+ \wedge (\Delta^n / \partial \Delta^n))}{\bigvee_{y \in L_n Y(\mathcal{U})} (F_{y(0)}(\mathcal{U})_+ \wedge (\Delta^n / \partial \Delta^n))},$$

and both Y_n and $L_n Y$ are presheaves. This presheaf is weakly equivalent in the global model structure to the cofiber C_n as shown by the following lemma.

Lemma 7.5. *There is a functorial weak equivalence in \mathbf{sSet}*

$$C_n(\mathcal{U}) \xleftarrow{\cong} \bigvee_{y \in N_n Y(\mathcal{U})} (F_{y(0)}(\mathcal{U})_+ \wedge (\Delta^n / \partial \Delta^n)).$$

Proof. Since $0 : * \rightarrow \Delta^n$ is a global weak equivalence, so is $F_y \rightarrow \Delta^n \times F_y$ for any $y : \Delta^n \rightarrow Y$, and the analogous statement holds for $y : \Delta^n \times \mathcal{U} \rightarrow Y$. For such $y : \Delta^n \times \mathcal{U} \rightarrow Y$, we obtain a map $\Delta^n \times \mathcal{U} \times F_{y(0)} \rightarrow \Delta^n \times \mathcal{U} \xrightarrow{y} Y$, which gives the bottom horizontal map in a commutative diagram of solid arrows in

$$\begin{array}{ccc} F_{y(0)} \times \mathcal{U} & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \downarrow f \\ \Delta^n \times \mathcal{U} \times F_{y(0)} & \longrightarrow & Y, \end{array}$$

whose top horizontal map is the inclusion of the fiber. Since f is a global fibration, we have the dotted arrow $\Delta^n \times \mathcal{U} \times F_{y(0)} \rightarrow X$ keeping the diagram commutative, which determines a commutative diagram

$$\begin{array}{ccc} \Delta^n \times \mathcal{U} \times F_{y(0)} & \longrightarrow & X \times_Y (\Delta^n \times \mathcal{U}) \\ \downarrow & & \downarrow \\ \Delta^n \times \mathcal{U} & \xrightarrow{\text{id}} & \Delta^n \times \mathcal{U}. \end{array}$$

Since both vertical arrows are global fibrations and the induced maps on the fibers and base are global weak equivalences, we have that $\Delta^i \times F_{y(0)} \rightarrow X \times_Y \Delta^i$ is a global equivalence, as can be seen by evaluating on $\mathcal{U} \in \mathbf{Sm}_k$. (Here it is implicitly used that a global fibration evaluated at \mathcal{U} is a fibration in \mathbf{sSet} , as can be seen by applying the defining lifting criterium to maps $\mathcal{U} \times Z_1 \rightarrow \mathcal{U} \times Z_2$ for $Z_1 \rightarrow Z_2$ a trivial cofibration in \mathbf{sSet} .)

Let $\partial \Delta^n$ denote the boundary of Δ^n . By [GJ09, VII Proposition 1.7 p. 355], there is a push-out

$$(20) \quad \begin{array}{ccc} (Y_n \times \partial \Delta^n) \cup_{L_n Y \times \partial \Delta^n} (L_n Y \times \Delta^n) & \longrightarrow & \text{sk}_{n-1} Y \\ \downarrow & & \downarrow \\ Y_n \times \Delta^n & \longrightarrow & \text{sk}_n Y \end{array}$$

The pull-back of a push-out square of simplicial sets is a push-out square because small colimits are pull-back stable in the topos of simplicial sets. Since limits and colimits in \mathbf{sPre} commute with taking the sections above $\mathcal{U} \in \mathbf{Sm}$, it follows that the pull-back of a push-out square in \mathbf{sPre} is also a push-out square. Thus applying the functor $(-)\times_Y X$ to (20) produces a push-out square.

We showed above that for any map $y : \Delta^n \rightarrow Y$, the pull-back $\Delta^n \times_Y X$ is globally equivalent to the trivial fibration $\Delta^n \times F_{y(0)}$. It follows that there is a global equivalence between C_n and the cofiber of

$$\left(\coprod_{y \in Y_n} \partial \Delta^n \times F_{y(0)} \right) \cup_{\coprod_{y \in L_n Y} \partial \Delta^n \times F_{y(0)}} \left(\coprod_{y \in L_n Y} \Delta^n \times F_{y(0)} \right) \rightarrow \coprod_{y \in Y_n} \Delta^n \times F_{y(0)}.$$

For typographical considerations, let's note that if you choose an equivalence between the various $F_{y(0)}$ and a fixed fiber F , this map is written

$$(Y_n \times \partial \Delta^n \times F) \cup_{L_n Y \times \partial \Delta^n \times F} (L_n Y \times \Delta^n \times F) \rightarrow Y_n \times \Delta^n \times F.$$

Even without such choices, this cofiber evaluated at \mathcal{U} is

$$\bigvee_{y \in (Y_n - L_n Y)(\mathcal{U})} F_{y(0)}(\mathcal{U})_+ \wedge (\Delta^n / \partial \Delta^n),$$

proving the lemma. \square

Lemma 7.6. *The spectral sequence of Definition 7.3 satisfies the property that $E_{ij}^r = 0$ for i or j less than 0. In particular, this spectral sequence converges.*

Proof. The claim is immediate for $i < 0$. We show that $E_{ij}^1 = 0$ for $j < 0$, which is sufficient because E_{ij}^r is a subquotient of $E_{i,j}^1$. By Lemma 7.5, $C_i = S^i \wedge D_i$ where $D_i \in \mathbf{sPre}_*$ is defined

$$D_i(\mathcal{U}) = \bigvee_{y \in N_i Y(\mathcal{U})} (F_{y(0)}(\mathcal{U})_+).$$

By (3) of Proposition 7.1, it follows that

$$E_{i,j}^1 = \mathcal{H}_{i+j} C_i \cong \mathcal{H}_{i+j} S^i \wedge D_i \cong \mathcal{H}_j D_i.$$

For $j < 0$, we have $\mathcal{H}_j D_i = 0$ by (1) Assumption 7.2. \square

Proposition 7.7. *The spectral sequence of Definition 7.3 converges to the values of the functors \mathcal{H}_* on X*

$$(E_{ij}^r, d^r : E_{i+r,j}^r \rightarrow E_{i,j+r-1}^r) \Rightarrow \mathcal{H}_{i+j} X.$$

Proof. We show that applying \mathcal{H}_i to the map $\mathrm{sk}_{n+1} Y \times_Y X \rightarrow X$ is an isomorphism for all n and $i \leq n$. This is sufficient to prove the proposition by Lemma 7.6.

Fix n . To show that \mathcal{H}_i applied to the map $\mathrm{sk}_{n+1} Y \times_Y X \rightarrow X$ is an isomorphism for $i \leq n$, it is sufficient to show that its cofiber, C , satisfies the condition that $\mathcal{H}_i C = 0$ for $i \leq n+1$ by (2) Proposition 7.1.

Any simplicial set is the colimit of its skeleta sk_j , and since colimits commute with taking sections at $\mathcal{U} \in \mathbf{Sm}_k$, it follows that $Y = \mathrm{colim}_j \mathrm{sk}_j Y$. Since filtered colimits commute with pullbacks [ML98, IX.2 Thm 1], it follows that $X = \mathrm{colim}_j (\mathrm{sk}_j Y \times_Y X)$. Since colimits commute, it follows that $C = \mathrm{colim}_j ((\mathrm{sk}_j Y \times_Y X) / (\mathrm{sk}_{n+1} Y \times_Y X))$. By (2) Assumption 2, it thus suffices to see that $\mathcal{H}_i(\mathrm{sk}_j Y \times_Y X) / (\mathrm{sk}_{n+1} Y \times_Y X) = 0$ for $i \leq (n+1)$ and $j \geq n+1$.

Using induction on j , the cofiber sequences

$$(\mathrm{sk}_{j-1} Y \times_Y X) / (\mathrm{sk}_{n+1} Y \times_Y X) \rightarrow (\mathrm{sk}_j Y \times_Y X) / (\mathrm{sk}_{n+1} Y \times_Y X) \rightarrow C_j$$

for $j-1 \geq n+1$, and (2) Proposition 7.1, it suffices to show that $\mathcal{H}_i C_j = 0$ for $i \leq (n+1)$ and $j \geq n+1$. This follows by Lemma 7.5 and (3) of Proposition 7.1 and (1) of Assumption 7.2. \square

Proposition 7.8. *Suppose $Y \in \mathbf{sPre}$ is such that $Y_0 = *$ is the constant presheaf on the point. Let $f : X \rightarrow Y$ be a map, and let $F = \mathrm{hofib}_{\mathrm{global}} f$ be the homotopy fiber in the global model structure. Let $E_{i,j}^r$ be the spectral sequence of Definition 7.3 applied to a fibrant replacement of f in the global model structure. Then there is a canonical isomorphism $E_{i,j}^1 \cong \mathcal{H}_j D_i$ where $D_i \in \mathbf{sPre}$ is defined by $D_i(\mathcal{U}) = \bigvee_{N_i Y(\mathcal{U})} (F(\mathcal{U})_+)$.*

Proof. Factor f into $f = \iota \circ f'$, with $\iota : X \rightarrow Z$ a global trivial cofibration and $f' : Z \rightarrow Y$ a global fibration such that $E_{i,j}^r$ is the spectral sequence of Definition 7.3 applied to f' and F is the fiber of f' above $* \rightarrow Y$ determined by $Y_0 = *$. By definition $E_{i,j}^1 \cong \mathcal{H}_{i+j} C_i$, where C_i is the cofiber of

$$\mathrm{sk}_{i-1} Y \times_Y Z \rightarrow \mathrm{sk}_i Y \times_Y Z.$$

By Lemma 7.5, we have a canonical global equivalence from the presheaf $\mathcal{U} \mapsto \bigvee_{N_i Y(\mathcal{U})} (F(\mathcal{U})_+ \wedge (\Delta^i / \partial \Delta^i))$ to C_i . Thus $C_i \cong S^i \wedge D_i$ showing the result by (3) Proposition 7.1. \square

Remark 7.9. Note that the construction of the spectral sequence of Definition 7.3 as well as those of its properties given in this section only require the lifting properties of global fibrations. The subtler lifting properties for \mathfrak{a} -fibrations have not been exploited.

7.2. A functoriality property. We will use a functoriality property of the spectral sequence constructed in Section 7.1 with respect to a particular sort of stable map in the homotopy category.

Recall that \mathfrak{a} is a left Bousfield localization of the global injective model structure.

Proposition 7.10. *The model structure \mathfrak{a} on $\mathbf{sPre}(\mathbf{Sm}_k)$ satisfies the following properties:*

- (1) *All monomorphisms in \mathbf{sPre} are \mathfrak{a} cofibrations, and in particular, objects of \mathbf{sPre} are \mathfrak{a} cofibrant.*
- (2) *\mathfrak{a} is compatible with the tensor, cotensor and simplicial enrichment in the sense that this structure makes \mathbf{sPre} into a simplicial model category [Rie14, Definition 11.4.4].*
- (3) *$\Sigma : \mathbf{sPre} \rightarrow \mathbf{sPre}$ takes \mathfrak{a} weak equivalences to \mathfrak{a} weak equivalences.*
- (4) *If X is an object of $\mathbf{sPre}(\mathbf{Sm}_k)$, then $X \times \cdot$ preserves weak equivalences.*
- (5) *There is a left Quillen functor $\Sigma^\infty : \mathbf{sPre}(\mathbf{Sm}_k)_+ \rightarrow \mathbf{Spt}(\mathbf{Sm}_k)$, where $\mathbf{Spt}(\mathbf{Sm}_k)$ is endowed with a stable model structure which we also call \mathfrak{a} , in an abuse of notation.*

Proof. (1) is immediate: \mathfrak{a} is a left Bousfield localization of the global injective model structure, and the same property holds there. (2) follows from [Hir03, Theorem 4.1.1(4)] because the global injective model structure is left proper, simplicial, and cellular (see [Hor06]). (3) is a special case of (2). For (4), see Lemma 2.1. (5) is Proposition 2.13. \square

To construct the EHP fiber sequence, we will use \mathfrak{a} to be the P -localized \mathbb{A}^1 -model structure for P a set of primes and $\mathcal{H}_i = \pi_i^{s,P,\mathbb{A}^1}$. As commented in Section 7.1, these choices are valid.

Let $j : J \rightarrow Y$ be a map of pointed simplicial presheaves, and let $b : \Sigma J \rightarrow \Sigma X$ be a map in $\mathrm{ho}_{\mathfrak{a}} \mathbf{sPre}$ between the suspensions of J and X , for X a pointed simplicial presheaf. Factor j as $j = \iota \circ f$ with $\iota : J \rightarrow E$ an \mathfrak{a} weak equivalence and cofibration, and $f : E \rightarrow Y$ an \mathfrak{a} fibration. Note since \mathfrak{a} is a left Bousfield localization of the global injective model structure, f is in particular a global fibration.

We use b and j to construct a map $\Sigma E \rightarrow \Sigma(X \times Y)$ in $\mathrm{ho}_{\mathfrak{a}} \mathbf{sPre}$ as follows. b induces a map $b_+ : \Sigma J_+ \rightarrow \Sigma X_+$ in $\mathrm{ho}_{\mathfrak{a}} \mathbf{sPre}$ between the suspensions of $J_+ = J \vee S^0$ and $X_+ = X \vee S^0$. The suspension of an \mathfrak{a} weak equivalence is an \mathfrak{a} weak equivalence by (3) Proposition 7.10. Thus, precomposing b_+ with $\Sigma E_+ \leftarrow \Sigma J_+$ defines a map $b_E : \Sigma E_+ \rightarrow \Sigma X_+$ in $\mathrm{ho}_{\mathfrak{a}} \mathbf{sPre}$. Since $E_+ \wedge (-) : \mathbf{sPre} \rightarrow \mathbf{sPre}$ preserves \mathfrak{a} weak equivalences by Corollary 2.2, we obtain a map

$$b_E \wedge \mathrm{id}_{E_+} : \Sigma(E \times E)_+ \rightarrow \Sigma(X \times E)_+$$

in $\mathrm{ho}_{\mathfrak{a}} \mathbf{sPre}$. The map $f : E \rightarrow Y$ induces a map $\Sigma(X \times E)_+ \rightarrow \Sigma(X \times Y)_+$ in \mathbf{sPre} . Composing with $b_E \wedge \mathrm{id}_{E_+}$ we obtain

$$b_E \wedge f_+ : \Sigma(E \times E)_+ \rightarrow \Sigma(X \times Y)_+$$

in $\mathrm{ho}_{\mathfrak{a}} \mathbf{sPre}$. The diagonal of E induces a map $\Sigma \Delta_+ : \Sigma E_+ \rightarrow \Sigma(E \times E)_+$. Composing with the previous yields a map

$$(b_E \wedge f_+) \circ \Sigma \Delta_+ : \Sigma E_+ \rightarrow \Sigma(X \times Y)_+$$

in $\mathrm{ho}_{\mathfrak{a}} \mathbf{sPre}$. This map in turn produces a map $\Sigma E \rightarrow \Sigma(X \times Y)$ in $\mathrm{ho}_{\mathfrak{a}} \mathbf{sPre}$ by including ΣE into ΣE_+ (this map does not respect base points) and mapping $\Sigma(X \times Y)_+$ to $\Sigma(X \times Y)$ by crushing the base point of $(X \times Y)_+$ to the base point of $X \times Y$ defined from the base points of X and Y .

Let $p : X \times Y \rightarrow Y$ denote the projection. The global fibrations f and p give rise to spectral sequences as in Definition 7.3. We show that $(b_E \wedge f_+) \circ \Sigma \Delta_+$ induces a map from the spectral sequence E_{ij}^r associated to f to the spectral sequence $(E')_{ij}^r$ associated to p .

For this, note that there are maps in $\text{ho}_a \mathbf{sPre}$

$$\Sigma(\text{sk}_n Y \times_Y E)_+ \rightarrow \Sigma(X \times \text{sk}_n Y)_+$$

constructed in a manner closely analogous to the definition of $(b_E \wedge f_+) \circ \Sigma\Delta_+ : \Sigma E_+ \rightarrow \Sigma(X \times Y)_+$. Explicitly, the suspension of the monomorphism $(\text{sk}_n Y \times_Y E)_+ \rightarrow E_+$ composed with b_E produces a map $b_{E_n} : \Sigma(\text{sk}_n Y \times_Y E)_+ \rightarrow \Sigma X_+$ in $\text{ho}_a \mathbf{sPre}$. Since $(\text{sk}_n Y \times_Y E)_+ \wedge (-) : \mathbf{sPre} \rightarrow \mathbf{sPre}$ preserves a weak equivalences by Corollary 2.2, we obtain a map

$$b_{E_n} \wedge \text{id}_{(\text{sk}_n Y \times_Y E)_+} : \Sigma((\text{sk}_n Y \times_Y E) \times (\text{sk}_n Y \times_Y E))_+ \rightarrow \Sigma(X \times (\text{sk}_n Y \times_Y E))_+$$

in $\text{ho}_a \mathbf{sPre}$. Precomposing with a map induced from the diagonal of $(\text{sk}_n Y \times_Y E)$ and post-composing with the map $\Sigma(X \times (\text{sk}_n Y \times_Y E))_+ \rightarrow \Sigma(X \times \text{sk}_n Y)_+$ induced from the map $f : (\text{sk}_n Y \times_Y E) \rightarrow \text{sk}_n Y$, we obtain the claimed map

$$b_{E_n} \wedge f_+ \circ \Sigma\Delta_+ : \Sigma(\text{sk}_n Y \times_Y E)_+ \rightarrow \Sigma(X \times \text{sk}_n Y)_+.$$

These maps are compatible in the sense that the diagram

$$(21) \quad \begin{array}{ccccc} \Sigma(\text{sk}_{n-1} Y \times_Y E)_+ & \longrightarrow & \Sigma(\text{sk}_n Y \times_Y E)_+ & \longrightarrow & \Sigma E_+ \\ \downarrow (b_{E_{n-1}} \wedge f_+) \circ \Sigma\Delta_+ & & \downarrow (b_{E_n} \wedge f_+) \circ \Sigma\Delta_+ & & \downarrow (b_E \wedge \Sigma f_+) \circ \Sigma\Delta_+ \\ \Sigma(\text{sk}_{n-1} Y \times X)_+ & \longrightarrow & \Sigma(\text{sk}_n Y \times X)_+ & \longrightarrow & \Sigma(Y \times X)_+ \end{array}$$

commutes.

By the assumption that \mathcal{H}_i factors through $\Sigma^\infty : \mathbf{sPre} \rightarrow \mathbf{Spt}(\mathbf{Sm}_k)$ and (3) Proposition 7.1 we may apply $\mathcal{H}_i \Sigma^{-1}$ to a map g in $\text{ho}_a \mathbf{sPre}$; indeed the resulting map can be identified with $\mathcal{H}_{i+1}(g)$.

We will use the following functoriality of the spectral sequence 7.3 with respect to the map

$$(22) \quad b : \Sigma E \rightarrow \Sigma(X \times Y)$$

induced from the composition $(b_E \wedge f_+) \circ \Sigma\Delta_+$. (Note again that we can include or quotient to get rid of a disjoint base point.)

Lemma 7.11. *There is a map of spectral sequences $E_{ij}^r \rightarrow (E')_{ij}^r$ such that the induced map $E_{ij}^\infty \rightarrow (E')_{ij}^\infty$ (which by Proposition 7.7 is a map from the associated graded of a filtration of $\mathcal{H}_{i+j}(E)$ to the associated graded of a filtration of $\mathcal{H}_{i+j}(X \times Y)$) is compatible with*

$$\mathcal{H}_{i+j} \Sigma^{-1} b : \mathcal{H}_{i+j}(E) \rightarrow \mathcal{H}_{i+j}(X \times Y).$$

Indeed, we will see in the proof of Lemma 7.11 that it is fair to say that b induces the map of spectral sequences $E_{ij}^r \rightarrow (E')_{ij}^r$.

Proof. The commutative diagram (21) induces a commutative diagram

$$(23) \quad \begin{array}{ccccc} \Sigma(\text{sk}_{n-1} Y \times_Y E)_+ & \longrightarrow & \Sigma(\text{sk}_n Y \times_Y E)_+ & \longrightarrow & \Sigma C_n \\ \downarrow (b_{E_{n-1}} \wedge f_+) \circ \Sigma\Delta_+ & & \downarrow (b_{E_n} \wedge f_+) \circ \Sigma\Delta_+ & & \downarrow \\ \Sigma(\text{sk}_{n-1} Y \times X)_+ & \longrightarrow & \Sigma(\text{sk}_n Y \times X)_+ & \longrightarrow & \Sigma C'_n \end{array}$$

in $\text{ho}_a \mathbf{sPre}$, where the horizontal rows are cofiber sequences and suspensions of cofiber sequences. Applying $\mathcal{H}_* \Sigma^{-1}$ to each of the horizontal rows produces long exact sequences ((2) Proposition 7.1). Applying $\mathcal{H}_* \Sigma^{-1}$ to the entire diagram then defines a morphism of long exact sequences, and thus a morphism of exact couples, and a morphism of spectral sequences. The compatibility with the induced map on E^∞ pages follows from applying $\mathcal{H}_* \Sigma^{-1}$ to (21). \square

We can compute the map $E_{ij}^1 \rightarrow (E')_{ij}^1$ of E^1 -pages of the map of spectral sequences of Lemma 7.11. For simplicity, suppose again that $Y \in \mathbf{sPre}$ is such that $Y_0 = *$. Let $\alpha : F \rightarrow E$ denote the canonical map of simplicial presheaves given by the definition $F = \text{hofib}_{\text{global}} f = * \times_Y E \rightarrow E$. Composing $\Sigma\alpha$ with b yields a map $b \circ \Sigma\alpha : \Sigma F \rightarrow \Sigma X$ in $\text{ho}_a \mathbf{sPre}$.

Recall that $D_i \in \mathbf{sPre}$ is defined by $D_i(U) = \bigvee_{N_n Y(U)} (F(U)_+)$. The analogous definition for the global fibration p is then $D'_i = \bigvee_{N_n Y(U)} (X(U)_+)$. We claim that the map $b \circ \Sigma\alpha : \Sigma F \rightarrow \Sigma X$ in $\text{ho}_a \mathbf{sPre}$ defines a map $\Sigma D_i \rightarrow \Sigma D'_i$ in $\text{ho}_a \mathbf{sPre}$. This claim is established by the following lemma and remark.

Lemma 7.12. *Suppose I is a presheaf of sets on \mathbf{Sm}_k .*

- (1) *If $g : A \rightarrow B$ is α -weak equivalence in pointed spaces $\mathbf{sPre}_*(\mathbf{Sm}_k)$, then $\bigvee_I g : \bigvee_I A \rightarrow \bigvee_I B$ is an α -weak equivalence.*
- (2) *If $g : A \rightarrow B$ be an α -weak equivalence in spectra $\mathbf{Spt}(\mathbf{Sm}_k)$, then $\bigvee_I g : \bigvee_I A \rightarrow \bigvee_I B$ is an α -weak equivalence.*

Furthermore, suppose that $Y \in \mathbf{sPre}$ is pointed. Then we may replace the presheaf I of sets on \mathbf{Sm}_k by $U \mapsto N_n Y(U)$.

Proof. (1): $\bigvee_I g : \bigvee_I A \rightarrow \bigvee_I B$ is canonically identified with $I_+ \wedge g : I_+ \wedge A \rightarrow I_+ \wedge B$, so $\bigvee_I g$ is a weak equivalence by Corollary 2.2.

(2): We again have a canonical identification of $\bigvee_I g : \bigvee_I A \rightarrow \bigvee_I B$ with $I_+ \wedge g : I_+ \wedge A \rightarrow I_+ \wedge B$. For any pointed simplicial sheaf \mathcal{X} , the functor

$$\mathbf{Spt}(\mathbf{Sm}_k) \rightarrow \mathbf{Spt}(\mathbf{Sm}_k)$$

$$E \rightarrow E \wedge \mathcal{X}$$

preserves stable \mathbb{A}^1 -weak equivalences, as in [Mor05, §4 pg 27].

Furthermore, note that we have a canonical bijection $Y_n(U)/L_n Y(U) \cong N_n Y(U) \coprod *$ and that $Y_n(U)/L_n Y(U)$ is a presheaf of sets. It follows that $U \mapsto \bigvee_{N_n Y(U)} g(U)$ is a retract of $\bigvee_I g$, where I is the presheaf of sets $Y_n(U)/L_n Y(U)$. We apply the above to I and conclude that $\bigvee_I g$ is an \mathbb{A}^1 -weak equivalence. Since weak equivalences are closed under retracts in any model category, we conclude that $U \mapsto \bigvee_{N_n Y(U)} g(U)$ is a weak equivalence as claimed. \square

It follows that $b \circ \Sigma\alpha$ in $\text{ho}_a \mathbf{sPre}$ induces a map

$$\bigvee_{N_n Y} (b \circ \Sigma\alpha) : \Sigma D_i \rightarrow \Sigma D'_i$$

in $\text{ho}_a \mathbf{sPre}$. As before, we may apply $\mathcal{H}_j \Sigma^{-1}$ to any map in $\text{ho}_a \mathbf{sPre}$. Applying $\mathcal{H}_j \Sigma^{-1}$ to $\bigvee_{N_n Y} (b \circ \Sigma\alpha)$ gives our identification of the map of E^1 -pages in the map of spectral sequences $E_{ij}^1 \rightarrow (E')_{ij}^1$ in Lemma 7.11.

Lemma 7.13. *Suppose that $Y_0 = *$. The identification of $E_{i,j}^1$ with $\mathcal{H}_j(D_i)$ in Proposition 7.8 is functorial with respect to the map*

$$E_{ij}^1 \rightarrow (E')_{ij}^1$$

in the sense that the induced map for $r = 1$ is $\mathcal{H}_j \Sigma^{-1}(\bigvee_{N_n Y(U)} (b \circ \Sigma\alpha)) : \mathcal{H}_j(D_i) \rightarrow \mathcal{H}_j(D'_i)$.

Proof. By construction, the lemma is equivalent to the claim that the map $\mathcal{H}_{n+j} C_n \rightarrow \mathcal{H}_{n+j} C'_n$ given by applying $\mathcal{H}_{n+j} \Sigma^{-1}$ to the commutative digram (23) is identified with

$$\bigvee_{\text{id}_{N_n Y}} (b \circ \Sigma\alpha) \wedge S^n : \Sigma \bigvee_{N_n Y} F_+ \wedge S^n \rightarrow \Sigma \bigvee_{N_n Y} X_+ \wedge S^n$$

via equivalences given in Lemma 7.5.

Equivalences in the context of Lemma 7.5 are constructed by choosing a trivialization

$$\begin{array}{ccc} \coprod_{Y_n} F & \xrightarrow{\quad} & E \\ \downarrow & \searrow \phi & \downarrow f \\ \coprod_{Y_n} (F \times \Delta^n) & \xrightarrow{\quad} & Y \end{array}$$

Note that the product $X \times Y$ admits a canonical trivialization

$$\begin{array}{ccc} \coprod_{Y_n} X & \xrightarrow{\quad} & X \times Y \\ \downarrow & \nearrow & \downarrow f \\ \coprod_{Y_n} (X \times \Delta^n) & \xrightarrow{\quad} & Y \end{array}$$

Let $\Sigma p_X : \Sigma(X \times \Delta^n) \rightarrow \Sigma X$ denote Σ applied to the projection $X \times \Delta^n \rightarrow X$. We then have a map in $\text{ho}_a \mathbf{sPre}$

$$\Sigma p_X \circ (b \wedge f_+) \circ \Sigma \Delta_+ \circ \Sigma \phi : \Sigma(\coprod_{Y_n} (F \times \Delta^n)) \rightarrow \Sigma(\coprod_{Y_n} X).$$

Choose a homotopy $H : I \times \Delta^n \rightarrow \Delta^n$ between the identity and the constant map whose value is the vertex 0. Pulling back along the resulting map

$$\Sigma(\coprod_{Y_n} (F \times I \times \Delta^n)) \rightarrow \Sigma(\coprod_{Y_n} (F \times \Delta^n))$$

produces a map

$$\tilde{H} : \Sigma(\coprod_{Y_n} (F \times \Delta^n \times I)) \rightarrow \Sigma(\coprod_{Y_n} X)$$

in $\text{ho}_a \mathbf{sPre}$.

Using the tensor structure of $\mathbf{Spt}(\mathbf{Sm}_k)$ over \mathbf{sPre} and Corollary 2.2, we have the map

$$\tilde{H}_+ \wedge \text{id}_{\Delta^n} : \Sigma(\coprod_{Y_n} (F \times I \times \Delta^n \times \Delta^n))_+ \rightarrow \Sigma(\coprod_{Y_n} (X \times \Delta^n))_+$$

in $\text{ho}_a \mathbf{sPre}$. Precomposing with a map induced from the diagonal $\Delta^n \rightarrow \Delta^n \times \Delta^n$ yields a map

$$\overline{H} : \Sigma(\coprod_{Y_n} (F \times I \times \Delta^n))_+ \rightarrow \Sigma(\coprod_{Y_n} (X \times \Delta^n))_+$$

in $\text{ho}_a \mathbf{sPre}$, which also induces a map without the disjoint basepoints, and which we will also denote by \overline{H} .

Let $\overline{H}|_0 : \Sigma(\coprod_{Y_n} (F \times \Delta^n)) \rightarrow \Sigma(\coprod_{Y_n} (X \times \Delta^n))$ in $\text{ho}_a \mathbf{sPre}$ denote the pullback by the inclusion $0 \rightarrow I$.

Unwinding definitions, and using the fact that $(b_E \wedge f_+) \circ \Sigma \Delta_+$ (and its analogues $\Sigma(\text{sk}_n Y \times_Y E)_+ \rightarrow \Sigma(X \times \text{sk}_n Y)_+$) post-composed with Σp_+ is Σf_+ in $\text{ho}_a \mathbf{sPre}$ (i.e. our maps respect the projections to ΣY_+), one can see that the map $\Sigma C_n \rightarrow \Sigma C'_n$ in the commutative diagram (23) is identified via equivalences given in Lemma 7.5 with the map

$$\Sigma \vee_{N_n Y} (F_+ \wedge \Delta^n / \partial \Delta^n) \rightarrow \Sigma \vee_{N_n Y} (X_+ \wedge \Delta^n / \partial \Delta^n)$$

induced by $\overline{H}|_0$ via the commutative diagram

$$\begin{array}{ccc}
 \Sigma \coprod_{L_n Y} (F \times \Delta^n) \cup \coprod_{L_n Y} (F \times \partial \Delta^n) \coprod_{Y_n} (F \times \partial \Delta^n) & \xrightarrow{\overline{H}|_0} & \Sigma \coprod_{L_n Y} (X \times \Delta^n) \cup \coprod_{L_n Y} (X \times \partial \Delta^n) \coprod_{Y_n} (X \times \partial \Delta^n) \\
 \downarrow & & \downarrow \\
 \Sigma \coprod_{Y_n} (F \times \Delta^n) & \xrightarrow{\overline{H}|_0} & \Sigma \coprod_{Y_n} (X \times \Delta^n) \\
 \downarrow & & \downarrow \\
 \Sigma \vee_{N_n Y} (F_+ \wedge \Delta^n / \partial \Delta^n) & \xrightarrow{\quad} & \Sigma \vee_{N_n Y} (X_+ \wedge \Delta^n / \partial \Delta^n)
 \end{array}$$

It follows that H induces a homotopy between $\Sigma C_n \rightarrow \Sigma C'_n$ and the map

$$\Sigma \vee_{N_n Y} (F_+ \wedge \Delta^n / \partial \Delta^n) \rightarrow \Sigma \vee_{N_n Y} (X_+ \wedge \Delta^n / \partial \Delta^n)$$

induced by $\overline{H}|_0$. The map induced by $\overline{H}|_1$ is $\vee_{Y_n/L_n Y} ((b \circ \Sigma a) \wedge \text{id}_{\Delta^n / \partial \Delta^n})$ showing the result. \square

The description of the map of E^1 -pages given in Lemma 7.13 means that understanding the construction taking a map $g : A \rightarrow B$ in $\text{ho}_a \mathbf{sPre}$ to $\vee_{N_n Y} g : \vee_{N_n Y} A \rightarrow \vee_{N_n Y} B$ in $\text{ho}_a \mathbf{sPre}$ implies understanding the map of E^1 -pages. The next lemmas give some understanding of this construction $g \mapsto \vee_{N_n Y} g$ in the case where (a, \mathcal{H}_*) is $(\mathbb{A}^1, \pi_*^{s, \mathbb{A}^1})$ or $(P - \mathbb{A}^1, \pi_*^{s, P, \mathbb{A}^1})$. In this case, we will prove Lemmas 7.15 and 7.16. Lemma 7.16 will be used in the next section to understand the map of E^1 -pages. Lemma 7.15 is included because we feel its hypothesis is more natural (although it assumes that g is a map in \mathbf{sPre} and not in the homotopy category), and we do not wish to give too much importance to the more elaborate hypothesis of Lemma 7.16.

To show Lemmas 7.15 and 7.16, it is useful to remark the following:

Remark 7.14. If X is a simplicially n -connected spectra in the sense that $X \in \mathbf{Spt}(\mathbf{Sm}_k)$ satisfies $\pi_i^s X = 0$ for $i \leq n$, then Morel's stable connectivity theorem [Mor05] implies that $\pi_i^{s, \mathbb{A}^1} X = \pi_i^s L_{\mathbb{A}^1} X = 0$ for $i \leq n$. Because $\pi_i^{s, P, \mathbb{A}^1}$ is naturally isomorphic to $\mathbb{Z}_P \otimes \pi_i^{s, \mathbb{A}^1}$ by Proposition 3.24, it also follows that $\pi_i^{s, P, \mathbb{A}^1}(X) = 0$ for $i \leq n$, whence $L_P L_{\mathbb{A}^1} X$ is simplicially n -connected.

Lemma 7.15. *Let I be a presheaf of sets on \mathbf{Sm}_k , and $g : A \rightarrow B$ be a map in \mathbf{sPre}_* . Let (a, \mathcal{H}_i) be either $(\mathbb{A}^1, \pi_i^{s, \mathbb{A}^1})$ or $(P - \mathbb{A}^1, \pi_i^{s, P, \mathbb{A}^1})$. If g induces an isomorphism on \mathcal{H}_i for $i < n$, then $\vee_I g : \vee_I A \rightarrow \vee_I B$ induces an isomorphism on \mathcal{H}_i for $i < n - 1$ and a surjection for $i = n - 1$.*

Proof. Let C denote the \mathbb{A}^1 -homotopy cofiber of g (i.e. factor g as $g_1 \circ g_2$ with $g_2 : A \rightarrow B'$ a cofibration and $g_1 : B' \rightarrow B$ an \mathbb{A}^1 fibration and an \mathbb{A}^1 weak equivalence and let C be the cofiber of the cofibration g_2). Since g_2 is also a $P - \mathbb{A}^1$ cofibration, and $B' \rightarrow B$ is also a $P - \mathbb{A}^1$ weak equivalence, we have a long exact sequence

$$(24) \quad \rightarrow \mathcal{H}_{i+1} C \rightarrow \mathcal{H}_i A \xrightarrow{\mathcal{H}_i(g)} \mathcal{H}_i B \rightarrow \mathcal{H}_i C \rightarrow \dots$$

Since $\mathcal{H}_i(g)$ is an isomorphism for $i < n$, we have that $\text{Image}(\mathcal{H}_{i+1} C \rightarrow \mathcal{H}_i A) = 0$. Thus $\mathcal{H}_{i+1} C = \text{Ker}(\mathcal{H}_{i+1} C \rightarrow \mathcal{H}_i A)$. Since (24) is exact, $\text{Ker}(\mathcal{H}_{i+1} C \rightarrow \mathcal{H}_i A) = \text{Image}(\mathcal{H}_{i+1} B \rightarrow \mathcal{H}_{i+1} C)$

Since $\mathcal{H}_i(g)$ is an isomorphism for $i < n$, we have that $\mathcal{H}_i B \rightarrow \mathcal{H}_i C$ is the zero map. Thus for $i < n - 1$, we have that $\text{Image}(\mathcal{H}_{i+1} B \rightarrow \mathcal{H}_{i+1} C) = 0$, from which it follows that $\mathcal{H}_{i+1} C = 0$. In other words, $\mathcal{H}_i C = 0$ for $i < n$.

For any point, q^* , the spectrum $q^* L_a \Sigma^\infty C$ satisfies that condition that $\pi_i^s q^* L_a \Sigma^\infty C = 0$ for $i < n$ because $\pi_i^s L_a \Sigma^\infty C = \mathcal{H}_i C = 0$, and $q^* \pi_i^s L_a \Sigma^\infty C = \pi_i^s q^* L_a \Sigma^\infty C$ [Mor05, 2.2 p. 12].

Thus $\bigvee_{q^*I} q^* L_a \Sigma^\infty C$ satisfies that condition that $\pi_i^s \bigvee_{q^*I} q^* L_a \Sigma^\infty C = 0$ for $i < n$. Note that $\bigvee_{q^*I} q^* L_a \Sigma^\infty C = q^*(\bigvee_I L_a \Sigma^\infty C)$. Thus $q^* \pi_i^s \bigvee_I L_a \Sigma^\infty C = \pi_i^s q^*(\bigvee_I L_a \Sigma^\infty C) = 0$. Since q was arbitrary, we conclude that $\pi_i^s \bigvee_I L_a \Sigma^\infty C = \pi_i^s q^*(\bigvee_I L_a \Sigma^\infty C) = 0$.

By Remark 7.14, we conclude that

$$\mathcal{H}_i \bigvee_I L_{\mathbb{A}^1} C = 0 \text{ for } i < n.$$

By Lemma 7.12, the map $\bigvee_I C \rightarrow \bigvee_I L_a \Sigma^\infty C$ is an ms-weak equivalence, whence

$$(25) \quad \mathcal{H}_i \bigvee_I C = 0 \text{ for } i < n.$$

By definition, we have that $A \xrightarrow{g_2} B' \rightarrow C$ is such that g_2 is a cofibration, and C is the cofiber of g_2 . By the definition of the global, injective local, \mathbb{A}^1 , or P, \mathbb{A}^1 -model structures, $\bigvee_I A \xrightarrow{g_2} \bigvee_I B'$ is a cofibration with cofiber $\bigvee_I C$. The sequence

$$\bigvee_I A \xrightarrow{g_2} \bigvee_I B' \rightarrow \bigvee_I C$$

therefore gives rise to a long exact sequence in \mathcal{H}_i . By Lemma 7.12, this long exact sequence can be written

$$(26) \quad \rightarrow \mathcal{H}_{i+1} \bigvee_I C \rightarrow \mathcal{H}_i \bigvee_I A \xrightarrow{\mathcal{H}_i(g)} \mathcal{H}_i \bigvee_I B \rightarrow \mathcal{H}_i \bigvee_I C \rightarrow \dots$$

The lemma is proven by combining (26) and (25). □

Lemma 7.16. *Let (a, \mathcal{H}_i) be either $(\mathbb{A}^1, \pi_i^{s, \mathbb{A}^1})$ or $(P - \mathbb{A}^1, \pi_i^{s, P, \mathbb{A}^1})$. Suppose I is a presheaf of sets on \mathbf{Sm}_k . Let $g : A \rightarrow B$ be a map in $\text{ho}_a \mathbf{Spt}(\mathbf{Sm}_k)$ which induces an isomorphism on \mathcal{H} for $i < n$, and suppose that there is a map $h : B \rightarrow A$ in $\text{ho}_a \mathbf{Spt}(\mathbf{Sm}_k)$ such that $g \circ h = \text{id}_B$ in $\text{ho}_a \mathbf{Spt}(\mathbf{Sm}_k)$. Then $\bigvee_I g : \bigvee_I A \rightarrow \bigvee_I B$ induces an isomorphism on \mathcal{H}_i for $i < n$.*

Proof. Since $g \circ h = \text{id}_B$, we have $\mathcal{H}_i g \circ \mathcal{H}_i h = \text{id}_{\mathcal{H}_i B}$, whence $\mathcal{H}_i g$ is surjective.

Choose a representative $g' : A \rightarrow L_a B$ of g . (This is possible because A is a -cofibrant.) Let C denote the \mathbb{A}^1 -homotopy cofiber of g , i.e. factor g' as $g' = g_1 \circ g_2$ with $g_2 : A \rightarrow B'$ a cofibration and $g_1 : B' \rightarrow L_a B$ a fibration and a weak equivalence and let C be the cofiber of the a cofibration g_2 . There is a long exact sequence

$$(27) \quad \rightarrow \mathcal{H}_{i+1} C \rightarrow \mathcal{H}_i A \xrightarrow{\mathcal{H}_i(g)} \mathcal{H}_i B \rightarrow \mathcal{H}_i C \rightarrow \dots$$

Since $\mathcal{H}_i g$ is surjective, $\text{Image } \mathcal{H}_i B = 0$ in $\mathcal{H}_i^{\mathbb{A}^1, s} C$. Since (27) is exact, it follows that

$$(28) \quad \text{Ker}(\mathcal{H}_i C \rightarrow \mathcal{H}_{i-1} A) = 0$$

for all i .

Since $\mathcal{H}_i g$ is an isomorphism for $i < n$, the exact sequence (27) shows that $\text{Image}(\mathcal{H}_{i+1} C \rightarrow \mathcal{H}_i A) = 0$.

Combining with (28) shows that $\mathcal{H}_i C = 0$ for $i \leq n$.

For any point, q^* , the spectrum $q^* L_a \Sigma^\infty C$ satisfies that condition that $\pi_i^s q^* L_a \Sigma^\infty C = 0$ for $i \leq n$ because $\pi_i^s L_a \Sigma^\infty C = \mathcal{H}_i C = 0$, and $q^* \pi_i^s L_a \Sigma^\infty C = \pi_i^s q^* L_a \Sigma^\infty C$ [Mor05, 2.2 p. 12].

Thus $\bigvee_{q^*I} q^* L_a \Sigma^\infty C$ satisfies that condition that $\pi_i^s \bigvee_{q^*I} q^* L_a \Sigma^\infty C = 0$ for $i \leq n$. Note that $\bigvee_{q^*I} q^* L_a \Sigma^\infty C = q^*(\bigvee_I L_a \Sigma^\infty C)$. Thus $q^* \pi_i^s \bigvee_I L_a \Sigma^\infty C = \pi_i^s q^*(\bigvee_I L_a \Sigma^\infty C) = 0$. Since q was arbitrary, we conclude that $\pi_i^s \bigvee_I L_a \Sigma^\infty C = 0$ for $i \leq n$. By Remark 7.14, we conclude that

$$(29) \quad \mathcal{H}_i \bigvee_I L_a \Sigma^\infty C = 0 \text{ for } i \leq n.$$

By Lemma 7.12, the map $\bigvee_I C \rightarrow \bigvee_I L_a \Sigma^\infty C$ is an a -weak equivalence, whence

$$(30) \quad \mathcal{H}_i \bigvee_I C = 0 \text{ for } i \leq n.$$

By the definition of the P, \mathbb{A}^1 or \mathbb{A}^1 -model structure, $\bigvee_I A \xrightarrow{g_2} \bigvee_I B' \rightarrow \bigvee_I C$ is a cofiber sequence and thus produces a long exact sequence

$$(31) \quad \rightarrow \mathcal{H}_{i+1} \bigvee_I C \rightarrow \mathcal{H}_i \bigvee_I A \xrightarrow{\mathcal{H}_i(g)} \mathcal{H}_i \bigvee_I B \rightarrow \mathcal{H}_i \bigvee_I C \rightarrow \dots$$

(This uses Lemma 7.12 as before.)

The lemma is proven by combining (31) and (30). □

Corollary 7.17. *Suppose that $Y \in \mathbf{sPre}$ is pointed. Then we may replace the presheaf I of sets on \mathbf{Sm}_k by $U \mapsto N_n Y(U)$ in Lemmas 7.15 and 7.16.*

Proof. Note that we have a canonical bijection $Y_n(U)/L_n Y(U) \cong N_n Y(U) \coprod *$ and that $Y_n(U)/L_n Y(U)$ is a presheaf of sets. It follows that $U \mapsto \bigvee_{N_n Y(U)} g(U)$ is a retract of $\bigvee_I g$, where I is the presheaf of sets $Y_n(U)/L_n Y(U)$. We apply Lemmas 7.15 and 7.16 to I and conclude that $\bigvee_I g$ induces an isomorphism on \mathcal{H}_i for $i < q - 1$ and a surjection for $i = q - 1$, and (respectively) induces an isomorphism on \mathcal{H}_i for $i < q$. These properties are all detected by injectivity or surjectivity statements concerning \mathcal{H}_i applied to a homotopy class of map of simplicial presheaves. Since this injectivity and surjectivity are preserved under retracts, it follows that we have the corresponding conclusion for the map $U \mapsto \bigvee_{N_n Y(U)} g(U)$ as claimed. □

7.3. Cancellation property. Say that a spectral sequence $E_{i,j}^r$ is a *first quadrant spectral sequence* if the differential on the r th page is of bidegree $(-r, r+1)$ i.e. $d_{i,j}^r : E_{i,j}^r \rightarrow E_{i-r,j+(r-1)}^r$, and if $E_{i,j}^r$ satisfies the condition that $E_{i,j}^r = 0$ when i or j is less than 0. The following lemma is a straight-forward consequence of degree considerations, but we include the proof for completeness.

Lemma 7.18. *Suppose $\theta_{i,j}^r : E_{i,j}^r \rightarrow (E')_{i,j}^r$ is a map of first quadrant spectral sequences such that $\theta_{i,j}^1$ is an isomorphism for $j < q$. Then*

- (1) $\theta_{i,j}^r$ is injective when $j < q$.
- (2) $\theta_{i,j}^r$ is an isomorphism when $j + (r-1) - 1 < q$.
- (3) $\theta_{i,j}^r$ is an isomorphism when $j < q$ and $j + i \leq q$.

Proof. We prove the claim by induction on q . For $q = 0$, there is nothing to show. Suppose the claim holds for $q - 1$. Now induct on r , which we assume ≥ 2 . Suppose that the claim holds for $r - 1$.

(1): Choose i, j with $j < q$. By the inductive hypothesis on r , we have that $\theta_{i,j}^{r-1}$ is injective. Since $(d')_{i,j}^{r-1} \theta_{i,j}^{r-1} = \theta_{i-(r-1),j+(r-1)-1}^{r-1} d_{i,j}^{r-1}$, it follows that $\text{Ker } d_{i,j}^{r-1} \rightarrow \text{Ker } (d')_{i,j}^{r-1}$ is injective. It thus suffices to show that $\text{Image } d_{i+(r-1),j-((r-1)-1)}^{r-1} \rightarrow \text{Image } (d')_{i+(r-1),j-((r-1)-1)}^{r-1}$ is surjective (which is equivalent to being an isomorphism because $\theta_{i,j}^{r-1}$ is injective). Let $j' = j - ((r-1) - 1)$. Note that $j' + ((r-1) - 1) - 1 = j - 1 < q$. Thus by the inductive hypothesis (2) on r , $\theta_{i+(r-1),j-((r-1)-1)}^{r-1}$ is an isomorphism from which the desired surjectivity follows.

(2): Choose i, j such that $j + (r-1) - 1 < q$. Since $j + ((r-1) - 1) - 1 < q$, we have by induction that $\theta_{i,j}^{r-1}$ is an isomorphism. We show that $\theta_{i,j}^r$ is an isomorphism. For this, it suffices to show that the inclusions $\text{Ker } d_{i,j}^{r-1} \subseteq \text{Ker } (d')_{i,j}^{r-1}$ and $\text{Image } d_{i+(r-1),j-((r-1)-1)}^{r-1} \subseteq \text{Image } (d')_{i+(r-1),j-((r-1)-1)}^{r-1}$ are isomorphisms. Since $j - ((r-1) - 1) + ((r-1) - 1) - 1 = j - 1 < q$, by the inductive hypothesis, we have that

$\theta_{i+(r-1),j-((r-1)-1)}^{r-1}$ is an isomorphism. Thus $\text{Image } d_{i+(r-1),j-((r-1)-1)}^{r-1} \subseteq \text{Image } (d')_{i+(r-1),j-((r-1)-1)}^{r-1}$ is an isomorphism. Note that $d_{i,j}^{r-1}$ is a map $E_{i,j}^{r-1} \rightarrow E_{i-(r-1),j+((r-1)-1)}^{r-1}$ and similarly for $(d')^{r-1}$. Thus to show that $\text{Ker } d_{i,j}^{r-1} \subseteq \text{Ker } (d')_{i,j}^{r-1}$ is an isomorphism, it suffices to show that $\theta_{i-(r-1),j+((r-1)-1)}^{r-1}$ is injective. Since $j + (r-1) - 1 < q$, $\theta_{i-(r-1),j+((r-1)-1)}^{r-1}$ is injective by (1).

(3): Choose i, j such that $j < q$ and $j + i \leq q$. By (1), we have that $\theta_{i,j}^r$ is injective. We show surjectivity. By the inductive hypothesis, $\theta_{i,j}^{r-1}$ is an isomorphism. Thus it suffices to see that $\text{Ker } d_{i,j}^{r-1} \subseteq \text{Ker } (d')_{i,j}^{r-1}$ is an isomorphism. Note that if $r = 2$, then $\theta_{i,j}^1$ and $\theta_{i-1,j}^1$ are isomorphisms and $\theta_{i-1,j}^1 d_{i,j}^1 = (d')_{i,j}^1 \theta_{i,j}^1$, whence $\text{Ker } d_{i,j}^1 \subseteq \text{Ker } (d')_{i,j}^1$ is an isomorphism. So, we may assume that $r > 2$. Since $(d')_{i,j}^{r-1} \theta_{i,j}^{r-1} = \theta_{i-(r-1),j+(r-1)-1}^{r-1} d_{i,j}^{r-1}$, it suffices to see that $\theta_{i-(r-1),j+(r-1)-1}^{r-1}$ is injective. For $i - (r-1) < 0$, we have that $E_{i-(r-1),j+(r-1)-1}^{r-1} = 0$ so $\theta_{i-(r-1),j+(r-1)-1}^{r-1}$ is injective. Thus we may assume $i - (r-1) \geq 0$ or equivalently $r \leq i + 1$. Thus $j + (r-1) - 1 \leq j + (i + 1 - 1) - 1 = j + i - 1 \leq q - 1 < q$. Thus $\theta_{i-(r-1),j+(r-1)-1}^{r-1}$ is injective by (1). □

Let (a, \mathcal{H}_i) be either $(\mathbb{A}^1, \pi_i^{s, \mathbb{A}^1})$ or $(P - \mathbb{A}^1, \pi_i^{s, P, \mathbb{A}^1})$.

As in Section 7.2, let $j : J \rightarrow Y$ be a map of pointed simplicial presheaves, and let $b : \Sigma J \rightarrow \Sigma X$ be a map in $\text{ho}_a \mathbf{sPre}$ between the suspensions of J and X , for X a pointed simplicial presheaf. Factor j as $j = \iota \circ f$ with $\iota : J \rightarrow E$ a weak equivalence and cofibration, and $f : E \rightarrow Y$ an a fibration. Note that $p : X \times Y \rightarrow Y$ is also an a fibration.

Let $b : \Sigma E \rightarrow \Sigma(X \times Y)$ in $\text{ho}_a \mathbf{sPre}$ be as in (22), i.e. b is the map induced from $b_E \wedge f_+ \circ \Sigma \Delta_+$, where $b_E : \Sigma E \rightarrow \Sigma X$ is the composition of b with $\Sigma E \leftarrow \Sigma J$, and Δ is the diagonal on E . See Section 7.2 for the construction. We view $\Sigma^{-1}b$ as a map from the fibration f to the fibration p .

Suppose that $Y_0 = *$. Let $a : F \rightarrow E$ denote the canonical map of simplicial presheaves $F = * \times_Y E \rightarrow E$.

Assume that $\mathcal{H}_0(X), \mathcal{H}_0(F), \mathcal{H}_0(Y)$, and $\mathcal{H}_0(E)$ are all 0. For example, this is satisfied if X, F, Y , and E are \mathbb{A}^1 -connected, because $\pi_i^{s, P, \mathbb{A}^1}(-) \cong \pi_i^{s, \mathbb{A}^1}(-) \otimes_{\mathbb{Z}} \mathbb{Z}_P$ – see Proposition 3.24.

Proposition 7.19. *If $\mathcal{H}_i(\Sigma^{-1}b)$ is an isomorphism for all i , then*

$$\mathcal{H}_i(\Sigma^{-1}b \circ a) : \mathcal{H}_i F \rightarrow \mathcal{H}_i X$$

is an isomorphism for all i , and $\Sigma^{-1}b \circ a : \Sigma^\infty F \rightarrow \Sigma^\infty X$ is an a weak equivalence.

Proof. Since (a, \mathcal{H}_i) is either $(\mathbb{A}^1, \pi_i^{s, \mathbb{A}^1})$ or $(P - \mathbb{A}^1, \pi_i^{s, P, \mathbb{A}^1})$, the functors \mathcal{H}_i detect stable a weak equivalences by Propositions 2.14, or 3.25 and 3.24. Thus it suffices to prove that $\mathcal{H}_i(\Sigma^{-1}b \circ a) : \mathcal{H}_i F \rightarrow \mathcal{H}_i X$ is an isomorphism for all i .

By Lemma 7.11, we have an induced morphism of spectral sequences $\theta_{i,j}^r : (E_{i,j}^r, d^r) \rightarrow ((E')_{i,j}^r, (d')^r)$ from the spectral sequence of Definition 7.3 induced by f to the one induced by p .

Suppose for the sake of contradiction that $\mathcal{H}_i(\Sigma^{-1}b \circ a) : \mathcal{H}_i F \rightarrow \mathcal{H}_i X$ is not an isomorphism for all i . Then there exists a minimal q such that $\mathcal{H}_q F \rightarrow \mathcal{H}_q X$ is not an isomorphism, and $q > 0$ by the assumption that both $\mathcal{H}_0 F$ and $\mathcal{H}_0 X$ are both 0. By Lemma 7.16 and Corollary 7.17, it follows that

$$(32) \quad \theta_{i,j}^1 : E_{i,j}^1 \xrightarrow{\cong} (E')_{i,j}^1$$

is an isomorphism for $j < q$.

By Lemmas 7.6 and 7.18, $\theta_{i,j}^r$ is an isomorphism for $i + j = q$ and $i > 0$ and all r . For r sufficiently large, $\theta_{i,j}^r = \theta_{i,j}^\infty$ (Lemma 7.6), whence $\theta_{i,j}^\infty$ is an isomorphism for $i + j = q$ and $i > 0$.

Introduce the notation

$$0 \subseteq R_n^0 \subseteq R_n^1 \subseteq R_n^2 \subseteq R_n^n = \mathcal{H}_n(E)$$

for the filtration associated to the spectral sequence $E_{i,j}^r$. Let the corresponding filtration associated to the spectral sequence $(E')_{i,j}^r$ be denoted $T_n^i \subseteq \mathcal{H}_n(X \times Y)$. Note that we have the commutative diagram

$$(33) \quad \begin{array}{ccccccc} 0 & \longrightarrow & T_q^i & \longrightarrow & T_q^{i+1} & \longrightarrow & (E')_{i+1,q-(i+1)}^\infty \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \theta_{i+1,q-(i+1)}^\infty \\ 0 & \longrightarrow & R_q^i & \longrightarrow & R_q^{i+1} & \longrightarrow & E_{i+1,q-(i+1)}^\infty \longrightarrow 0 \end{array}$$

for $i = -1, \dots, q-1$, where by convention $R_n^{-1} = 0$ and $T_n^{-1} = 0$ for all n .

Since $\Sigma^{-1}b$ induces an isomorphism on \mathcal{H}_q , we have that $R_q^q \rightarrow T_q^q$ is an isomorphism. Applying the 5-lemma and (33) for $i = q-1, q-2, \dots, 0$, we conclude that $R_q^0 \rightarrow T_q^0$ is an isomorphism. By definition, $R_q^0 = E_{0,q}^\infty$ and $T_q^0 = (E')_{0,q}^\infty$, so we have that

$$\theta_{0,q}^\infty : E_{0,q}^\infty \rightarrow (E')_{0,q}^\infty$$

is an isomorphism.

Since $Y_0 = *$, the map $\theta_{0,1}^1 : E_{0,q}^1 \rightarrow (E')_{0,q}^1$ is identified with $\mathcal{H}_q F \rightarrow \mathcal{H}_q X$ by Lemmas 7.8 and 7.13. Since we have assumed that $\mathcal{H}_q F \rightarrow \mathcal{H}_q X$ is not an isomorphism, it follows that

$$\theta_{0,1}^1 : E_{0,q}^1 \rightarrow (E')_{0,q}^1$$

is not an isomorphism. Let n be maximal such that $E_{0,q}^n \rightarrow (E')_{0,q}^n$ is not an isomorphism. Since $d_{0,q}^n = 0$, $(d')_{0,q}^n = 0$, and $E_{0,q}^{n+1} \rightarrow (E')_{0,q}^{n+1}$ is an isomorphism, we conclude that

$$\text{Image } d_{n,q-(n-1)}^n \rightarrow \text{Image } (d')_{n,q-(n-1)}^n$$

is not an isomorphism.

Note that $q+1-i+(r-1)-1 < q$ when $r < i+1$. Thus by Lemma 7.18 (2), we have that

$$\theta_{i,q+1-i}^r : E_{i,q+1-i}^r \rightarrow (E')_{i,q+1-i}^r$$

is an isomorphism for $i > 1$ and $r = 1, 2, \dots, i$. For $r = i+1$,

$$\begin{aligned} E_{i,q+1-i}^{i+1} &\cong \text{Ker } d_{i,q+1-i}^i / \text{Image } d_{2i,q+1-i-(i-1)}^i \\ (E')_{i,q+1-i}^{i+1} &\cong \text{ker } (d')_{i,q+1-i}^i / \text{Image } (d')_{2i,q+1-i-(i-1)}^i. \end{aligned}$$

Furthermore, we have the isomorphism $\text{Image } d_{2i,q+1-i-(i-1)}^i \cong \text{Image } (d')_{2i,q+1-i-(i-1)}^i$ by Lemma 7.18 (2). Thus $E_{i,q+1-i}^{i+1} \xrightarrow{\subseteq} (E')_{i,q+1-i}^{i+1}$ is an injection, and is an isomorphism if and only if

$$\text{Ker } d_{i,q+1-i}^i \rightarrow \text{Ker } (d')_{i,q+1-i}^i$$

is an isomorphism, which happens if and only if $\text{Image } d_{i,q+1-i}^i \rightarrow \text{Image } (d')_{i,q+1-i}^i$ is an isomorphism.

By the above, we thus conclude that $E_{n,q+1-n}^{n+1} \xrightarrow{\subseteq} (E')_{n,q+1-n}^{n+1}$ is not surjective.

Note that by degree reasons, $d_{n,q+1-n}^m = 0$ and $(d')_{n,q+1-n}^m = 0$ for $m \geq n+1$. Since by Lemma 7.18 (1) and (2) we have $\text{Image } d_{n+m,q+1-n-(m-1)}^m \cong \text{Image } (d')_{n+m,q+1-n-(m-1)}^m$, we conclude that

$$E_{n,q+1-n}^\infty \xrightarrow{\subseteq} (E')_{n,q+1-n}^\infty$$

is not surjective.

The same reasoning shows that $E_{i,q+1-i}^\infty \xrightarrow{\subseteq} (E')_{i,q+1-i}^\infty$ is an isomorphism for $i > n \geq 1$. Explicitly, for $i > n$, we have that $E_{0,q}^i \rightarrow (E')_{0,q}^i$ is an isomorphism by choice of n . As above, note that since $d_{0,q}^i = 0$, $(d')_{0,q}^i = 0$, and $E_{0,q}^{i+1} \rightarrow (E')_{0,q}^{i+1}$ is an isomorphism, we conclude that $\text{Image } d_{i,q-(i-1)}^i \rightarrow \text{Image } (d')_{i,q-(i-1)}^i$ is an isomorphism. Continuing with the same reasoning, note that $\theta_{i,q+1-i}^i : E_{i,q+1-i}^i \rightarrow (E')_{i,q+1-i}^i$ is an isomorphism by Lemma 7.18 (2), whence $\text{Ker } d_{i,q-(i-1)}^i \rightarrow \text{Ker } (d')_{i,q-(i-1)}^i$ is an isomorphism. Since $\text{Image } d_{2i,q+1-i-(i-1)}^i \cong \text{Image } (d')_{2i,q+1-i-(i-1)}^i$ by Lemma 7.18 (1) and (2), we have that $\theta_{i,q+1-i}^{i+1} : E_{i,q+1-i}^{i+1} \rightarrow (E')_{i,q+1-i}^{i+1}$ is an isomorphism. By degree reasons $d_{i,q+1-i}^m = 0$ and $(d')_{i,q+1-i}^m = 0$ for $m \geq i+1$. Since by Lemma 7.18 (1) and (2) we have $\text{Image } d_{i+m,q+1-i-(m-1)}^m \cong \text{Image } (d')_{i+m,q+1-i-(m-1)}^m$, we have that $E_{i,q+1-i}^\infty \xrightarrow{\subseteq} (E')_{i,q+1-i}^\infty$ is an isomorphism for $i > n$ as claimed.

Since $\Sigma^{-1}\mathfrak{b}$ induces an isomorphism on \mathcal{H}_{q+1} , we have that $R_{q+1}^{q+1} \rightarrow T_{q+1}^{q+1}$ is an isomorphism. Combining this with the commutative diagram

$$(34) \quad \begin{array}{ccccccc} 0 & \longrightarrow & T_{q+1}^i & \longrightarrow & T_{q+1}^{i+1} & \longrightarrow & (E')_{i+1,q+1-(i+1)}^\infty \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \theta_{i+1,q+1-(i+1)}^\infty \\ 0 & \longrightarrow & R_{q+1}^i & \longrightarrow & R_{q+1}^{i+1} & \longrightarrow & E_{i+1,q+1-(i+1)}^\infty \longrightarrow 0 \end{array}$$

for $i = q, q-1, \dots, n$ and the five lemma, we see that $R_{q+1}^i \rightarrow R_{q+1}^{i+1}$ is an isomorphism for $i = q, q-1, \dots, n$.

In particular, $R_{n+1}^n \rightarrow R_{q+1}^i$ is surjective, which by (34) with $i = n-1$ implies that $E_{n,q+1-n}^\infty \xrightarrow{\subseteq} (E')_{n,q+1-n}^\infty$ is surjective, giving the desired contradiction. \square

Here is a verbal description of the proof of Proposition 7.19. Choose q minimal such that $\pi_q^{s,\mathbb{A}^1}(F) \rightarrow \pi_q^{s,\mathbb{A}^1}(X)$ is not an isomorphism. The failure to be an isomorphism is necessarily a failure of injectivity. In terms of the map of spectral sequences, this implies that $E_{0,1}^1 \rightarrow (E')_{0,q}^1$ is not injective. Since q is minimal, degree arguments with first quadrant spectral sequences imply that $E_{i,q-i}^\infty \rightarrow (E')_{i,q-i}^\infty$ are isomorphisms for $i > 0$. Since $\pi_q^{s,\mathbb{A}^1}((b \wedge \Sigma^\infty f_+) \circ \Sigma^\infty \Delta_+)$ is an isomorphism, and since $\oplus_i E_{i,q-i}^\infty$ is the associated graded of $\pi_q^{\mathbb{A}^1,s}(E)$ and the analogous statement for E' and $X \times Y$ holds, it follows that $E_{0,q}^\infty \rightarrow (E')_{0,q-i}^\infty$ is also an isomorphism. Thus we can choose a maximal $n \geq 1$ for which $E_{0,q}^n \rightarrow (E')_{0,q}^n$ is not an isomorphism. For degree reasons, the failure of this map to be an isomorphism is a failure of injectivity. Thus the image of a d must be larger than the image of a (d') . Since the domains of these differentials have smaller second index, these domains actually have to be isomorphic. This then implies that the kernel of the d is smaller than the kernel of the (d') . This leads to a contradiction with the surjectivity of $\pi_{q+1}^{s,\mathbb{A}^1}(F) \rightarrow \pi_{q+1}^{s,\mathbb{A}^1}(X)$.

8. \mathbb{A}^1 SIMPLICIAL EHP FIBER SEQUENCE

Definition 8.1. Say that the sequence $X \rightarrow Y \rightarrow Z$ in $\mathbf{sPre}(\mathbf{Sm}_k)_*$ is a fiber sequence up to homotopy if there is a diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & E & \xrightarrow{f} & B \end{array}$$

which commutes up to homotopy with B fibrant, f a fibration with fiber $F = * \times_B E$, and all the vertical maps weak equivalences.

Remark 8.2. Morel defines $X \rightarrow Y \rightarrow Z$ to be a simplicial fibration sequence if the composition $X \rightarrow Z$ is the constant map and the induced map from X to the homotopy fiber of $Y \rightarrow Z$ in the injective local model structure is a simplicial weak equivalence.

He then defines $X \rightarrow Y \rightarrow Z$ to be an \mathbb{A}^1 -fibration sequence if $L_{\mathbb{A}^1} X \rightarrow L_{\mathbb{A}^1} Y \rightarrow L_{\mathbb{A}^1} Z$ is a simplicial fibration sequence. See [Mor12, Definition 6.44].

In this vane, it is natural to define $X \rightarrow Y \rightarrow Z$ to be a $P - \mathbb{A}^1$ fibration sequence if

$$L_P L_{\mathbb{A}^1} X \rightarrow L_P L_{\mathbb{A}^1} Y \rightarrow L_P L_{\mathbb{A}^1} Z$$

is a simplicial fibration sequence.

Note that if $X \rightarrow Y \rightarrow Z$ in $\mathbf{sPre}(\mathbf{Sm}_k)_*$ is a $P - \mathbb{A}^1$ fiber sequence up to homotopy as in 8.1, then B , E , and F are $P - \mathbb{A}^1$ local, from which it follows that they can be identified with $L_P L_{\mathbb{A}^1} X$, $L_P L_{\mathbb{A}^1} E$, and $L_P L_{\mathbb{A}^1} F$ respectively. Since $P - \mathbb{A}^1$ fibrations are simplicial fibrations, we have that a $P - \mathbb{A}^1$ fiber sequence up to homotopy is a $P - \mathbb{A}^1$ fibration sequence.

Note that for P the set of all primes, the $P - \mathbb{A}^1$ injective model structures on $\mathbf{sPre}(\mathbf{Sm}_k)$ and $\mathbf{Spt}(\mathbf{Sm}_k)$ are the \mathbb{A}^1 injective model structures, and $X \rightarrow L_P X$ is the identity map.

For $X \in \mathbf{sPre}$, let $j : J(X) \rightarrow J(X^{\wedge 2})$ be j_2 from Definition 5.5. Recall from Section 4.1 and before the notation $S^{n+q\alpha} = S^n \wedge \mathbb{G}_m^{\wedge q}$. Recall from Sections 1 and 6 the notation $-\langle -1 \rangle$ in $\mathrm{GW}(k)$. Recall from Section 3 that for a set of primes P , \mathbb{Z}_P denotes \mathbb{Z} with all primes not in P inverted.

Theorem 8.3. *Let $X = S^{n+q\alpha}$ with $n > 1$, and let $e = (-1)^{n+q} \langle -1 \rangle^q$. Let P be a set of primes, and suppose that for all $m \in \mathbb{Z}_{>0}$, the element $(m+1) + me$ is a unit in $\mathrm{GW}(k) \otimes \mathbb{Z}_P$. Then*

$$(35) \quad X \rightarrow J(X) \xrightarrow{j} J(X^{\wedge 2})$$

is a fiber sequence up to homotopy in the $P - \mathbb{A}^1$ injective model structure on $\mathbf{sPre}(\mathbf{Sm}_k)$.

By Proposition 5.2, Theorem 8.3 proves Theorem 1.3.

Proof. Let α denote the $P - \mathbb{A}^1$ injective model structure on $\mathbf{sPre}(\mathbf{Sm}_k)$ and $\mathbf{Spt}(\mathbf{Sm}_k)$.

Recall the notation $D(X) = \bigvee_{j=0}^{\infty} X^{\wedge j}$ from Definition 5.1. From Section (5), we have the zig-zag (12)

$$\Sigma J(X)_+ \longrightarrow \mathrm{Simp} |\Sigma D(X)| \longleftarrow \Sigma D(X)$$

of weak equivalences in the global model structure.

Let $b_1 : D(X) \rightarrow X$ denote the map which for $j \neq 1$ crushes the summands $X^{\wedge j}$ to the base point.

Replace $J(X^{\wedge 2})$ by a fibrant simplicial presheaf $J(X^{\wedge 2}) \rightarrow L_P L_{\mathbb{A}^1} J(X^{\wedge 2})$. Let $f' : E' \rightarrow L_P L_{\mathbb{A}^1} J(X^{\wedge 2})$ be a fibrant replacement of $J(X) \xrightarrow{j} J(X^{\wedge 2}) \rightarrow L_P L_{\mathbb{A}^1} J(X^{\wedge 2})$ is the α model structure. Let

$$E = J(X^{\wedge 2}) \times_{L_P L_{\mathbb{A}^1} J(X^{\wedge 2})} E'$$

be the pull-back of E' to $J(X^{\wedge 2})$, and let $f : E \rightarrow J(X^{\wedge 2})$ be the canonical projection. Note that f is an α fibration, and that there is a canonical map $J(X) \rightarrow E$. Since α is a proper model structure and $J(X^{\wedge 2}) \rightarrow L_P L_{\mathbb{A}^1} J(X^{\wedge 2})$ is a α weak equivalence, we have that $E \rightarrow E'$ is a weak equivalence. Since $J(X) \rightarrow E'$ is a α -weak equivalence by construction, by the 2-out-of-3 property, it follows that the canonical map $J(X) \rightarrow E$

is a α weak equivalence. We thus have a commutative diagram

$$\begin{array}{ccc} J(X) & \xrightarrow{\cong_\alpha} & E \\ & \searrow j & \downarrow f \\ & & J(X^{\wedge 2}) \end{array}$$

with the map $J(X) \xrightarrow{\cong} E$ a α -weak equivalence.

It follows that we have the zig-zag

$$\Sigma E \xleftarrow{\cong} \Sigma J(X)_+ \xrightarrow{\cong} \text{Simp } |\Sigma D(X)| \xleftarrow{\cong} \Sigma D(X) \xrightarrow{b_1} \Sigma X$$

which determines a map $b : \Sigma^\infty E \rightarrow \Sigma^\infty X$ in $\text{ho}_\alpha \mathbf{Spt}(\mathbf{Sm}_k)$.

As in Section 6.4 and 7.2, we obtain a map

$$(b_+ \wedge \Sigma^\infty f_+) \circ \Sigma^\infty \Delta_+ : \Sigma^\infty E_+ \rightarrow \Sigma^\infty (X \times Y)_+.$$

in $\text{ho}_\alpha \mathbf{Spt}(\mathbf{Sm}_k)$. The hypothesis that for all $m \in \mathbb{Z}_{>0}$, the element $((m+1) + me)$ is a unit in $\text{GW}(k) \otimes \mathbb{Z}_p$ implies that $(b_+ \wedge \Sigma^\infty f_+) \circ \Sigma^\infty \Delta_+$ is an isomorphism by Proposition 6.16. By Proposition 7.19, it follows that $b \circ \Sigma^\infty \alpha : \Sigma^\infty F \rightarrow \Sigma^\infty X$ is an isomorphism in $\text{ho}_\alpha \mathbf{Spt}(\mathbf{Sm}_k)$, where $\alpha : F \rightarrow E$ is the inclusion of the fiber of f into E .

Since E is the pull-back of E' we have the diagram in $\mathbf{sPre}(\mathbf{Sm}_k)$

$$\begin{array}{ccccc} F & \longrightarrow & E' & \longrightarrow & L_P L_{\mathbb{A}^1} J(X^{\wedge 2}) \\ \uparrow & & \uparrow \cong_\alpha & & \uparrow \cong_\alpha \\ F & \longrightarrow & E & \longrightarrow & J(X^{\wedge 2}) \end{array}$$

We conclude that $b \circ \Sigma^\infty \alpha' : \Sigma^\infty F \rightarrow \Sigma^\infty X$ is an isomorphism in $\text{ho}_\alpha \mathbf{Spt}(\mathbf{Sm}_k)$, where α' is the composition in $\text{ho}_\alpha \mathbf{Spt}(\mathbf{Sm}_k)$ corresponding to the zig-zag $F \rightarrow E' \leftarrow E$.

Since the composition of the two maps in the sequence (35) is constant, the composition

$$X \rightarrow J(X) \rightarrow L_P L_{\mathbb{A}^1} J(X^{\wedge 2})$$

is also constant. We therefore have an induced map

$$h : X \rightarrow .F$$

By construction of b_1 , the composition $\Sigma^\infty X \rightarrow \Sigma^\infty E \xrightarrow{b} \Sigma^\infty X$ is the identity in $\text{ho}_\alpha \mathbf{Spt}(\mathbf{Sm}_k)$. Thus the composition $\Sigma^\infty X \xrightarrow{\Sigma^\infty h} \Sigma^\infty F \xrightarrow{\Sigma^\infty \alpha'} \Sigma^\infty E \xrightarrow{b} \Sigma^\infty X$ is the identity. Since $b \circ \Sigma^\infty \alpha'$ is an isomorphism in $\text{ho}_\alpha \mathbf{Spt}(\mathbf{Sm}_k)$, so is $\Sigma^\infty h$.

Note that $L_P h : L_P X \rightarrow L_P F$ is a map of \mathbb{A}^1 -simply connected objects. $\Sigma^\infty L_P h$ is an \mathbb{A}^1 weak equivalence as follows. By Corollary 3.19, we have that $\Sigma^\infty L_P h \simeq L_P \Sigma^\infty h$. Since $\Sigma^\infty h$ is a $P\text{-}\mathbb{A}^1$ weak equivalence, we have that $L_P L_{\mathbb{A}^1} \Sigma^\infty h$ is a simplicial weak equivalence. By Proposition 3.23, we have $L_P L_{\mathbb{A}^1} \Sigma^\infty h \simeq L_{\mathbb{A}^1} L_P \Sigma^\infty h$, whence $L_{\mathbb{A}^1} L_P \Sigma^\infty h$ is a simplicial weak equivalence so $L_P \Sigma^\infty h$ is an \mathbb{A}^1 weak equivalence as claimed. We now apply Corollary 2.22 to conclude that $L_P h$ is an \mathbb{A}^1 weak equivalence, whence h is an α -weak equivalence.

The diagram

$$\begin{array}{ccccc} X & \longrightarrow & J(X) & \xrightarrow{j} & J(X^{\wedge 2}) \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & E' & \longrightarrow & L_P L_{\mathbb{A}^1} J(X^{\wedge 2}) \end{array}$$

shows that (35) is an a fiber sequence up to homotopy. \square

Proof. (Corollary 1.4)

- By Theorem 8.3, it is sufficient to show that $(m+1) + me$ is a unit in $GW(k) \otimes \mathbb{Z}_{(2)}$ for all positive integers m . This was shown in Corollary 4.7.
- Apply Theorem 8.3 and Corollary 4.8.
- When n is odd and q is even, we have $e = -1$, whence $((m+1) + me) = 1$ which is a unit in $GW(k)$.
- When $2\eta = 0$, we have $2\eta\rho = 0$, whence $\eta\rho$ is torsion in $GW(k)$. All torsion elements of $GW(k)$ are nilpotent [Lam05, Chapter VIII, §8.1] (or because the Grothendieck-Witt ring is a λ -ring, and all torsion elements of λ rings are nilpotent by a result of Graeme Segal), so $\eta\rho$ is nilpotent. When $n+q$ is odd,

$$e = (-1)^n (-1)^q (1 + \rho\eta)^q = -(1 + \rho\eta)^q = \begin{cases} -1 - \rho\eta & \text{if } q \text{ odd} \\ -1 & \text{if } q \text{ even.} \end{cases}$$

Thus, $1 + e$ is nilpotent, from which it follows that $((m+1) + me) = 1 + (e+1)m$ is a unit in $GW(k)$.

The fact that $2\eta = 0$ when $k = \mathbb{C}$ is shown [Mor04, Remark 6.3.5, Lemma 6.3.7]. \square

Corollary 8.4. *Let $X = S^{n+q\alpha}$ with $n > 1$. Choose $v \in \mathbb{Z}$. There is a functorial long exact sequence*

$$\dots \rightarrow \mathbb{Z}_{(2)} \otimes \pi_{i+v\alpha}^{\mathbb{A}^1} X \rightarrow \mathbb{Z}_{(2)} \otimes \pi_{i+1+v\alpha}^{\mathbb{A}^1} \Sigma X \rightarrow \mathbb{Z}_{(2)} \otimes \pi_{i+1+v\alpha}^{\mathbb{A}^1} \Sigma(X \wedge X) \rightarrow \mathbb{Z}_{(2)} \otimes \pi_{i-1+v\alpha}^{\mathbb{A}^1} X \rightarrow \dots$$

Proof. Combining Theorem 8.3 and Corollary 1.4, we have that

$$X \rightarrow J(X) \rightarrow J(X^{\wedge 2})$$

is a $2\text{-}\mathbb{A}^1$ fiber sequence up to homotopy. It follows that there is an associated long exact sequence in $\pi_{*+v\alpha}^{2, \mathbb{A}^1}$. See Proposition 3.16. By Proposition 3.15 and [Mor12, Theorem 6.13], we may replace $\pi_{*+v\alpha}^{2, \mathbb{A}^1}$ with $\mathbb{Z}_{(2)} \otimes \pi_{*+v\alpha}^{\mathbb{A}^1}$. By Corollary 5.3, we can identify the homotopy groups $\pi_{*+v\alpha}^{\mathbb{A}^1} J(X)$ with $\pi_{*+1+v\alpha}^{\mathbb{A}^1} \Sigma X$. This yields the claimed long exact sequence. \square

The long exact sequences of Corollary 8.4 form an exact couple, which in turn gives rise to an \mathbb{A}^1 simplicial EHP spectral sequence. Here the adjective “simplicial” refers to the suspension with respect to the simplicial circle S^1 .

Theorem 8.5. *Choose $q, v \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$ such that $n \geq 2$. There is a spectral sequence*

$$(E_{i,m}^r, d_r : E_{i,m}^r \rightarrow E_{i-1,m-r}^r) \Rightarrow \mathbb{Z}_{(2)} \otimes \pi_{i-n, v-q}^{s, \mathbb{A}^1} = \mathbb{Z}_{(2)} \otimes \pi_{i+v\alpha}^{s, \mathbb{A}^1} S^{n+q\alpha}$$

with $E_{i,m}^1 = \mathbb{Z}_{(2)} \otimes \pi_{m+1+i+v\alpha}^{\mathbb{A}^1} (S^{2m+2n+1+2q\alpha})$ if $i \geq 2n-1+m$ and otherwise $E_{i,m}^1 = 0$.

Proof. The E^1 page is as claimed by the construction of the exact couple and by the fact that $\pi_{i+v\alpha}^{\mathbb{A}^1} S^{n+q\alpha} = 0$ for $i < n$. The latter fact follows from Morel’s connectivity theorem [Mor12, Theorem 6.38].

The spectral sequence converges for degree reasons; for all (i, m) there are only finitely many r with a non-zero differential leaving or entering $E_{i,m}^r$.

Since by definition we have $\operatorname{colim}_m \pi_{m+i+v\alpha}^{\mathbb{A}^1} \Sigma^m S^{n+q\alpha} = \pi_{i+v\alpha}^{s, \mathbb{A}^1} S^{n+q\alpha}$ (see Section 2.4), we also have

$$\operatorname{colim}_m (\mathbb{Z}_{(2)} \otimes \pi_{i+m+v\alpha}^{\mathbb{A}^1} \Sigma^m S^{n+q\alpha}) = \mathbb{Z}_{(2)} \otimes \pi_{i+v\alpha}^{s, \mathbb{A}^1} S^{n+q\alpha}$$

and it follows that the spectral sequence converges to $\mathbb{Z}_{(2)} \otimes \pi_{i+v\alpha}^{s, \mathbb{A}^1} S^{n+q\alpha}$. \square

As in the setting of algebraic topology, one obtains truncated EHP spectral sequences converging to unstable homotopy groups of spheres.

Theorem 8.6. *Choose $q, v \in \mathbb{Z}_{\geq 0}$ and $n_1, n_2 \in \mathbb{Z}$ such that $n_2 \geq n_1 \geq 2$. There is a spectral sequence $(E_{i,m}^r, d_r : E_{i,m}^r \rightarrow E_{i-1,m-r}^r) \Rightarrow \mathbb{Z}_{(2)} \otimes \pi_{i+v\alpha}^{\mathbb{A}^1} S^{n_2+q\alpha}$ with*

$$E_{i,m}^1 = \begin{cases} \mathbb{Z}_{(2)} \otimes \pi_{m+1+i+v\alpha}^{\mathbb{A}^1} (S^{2m+2n_1+1+2q\alpha}) & \text{if } i \geq 2n_1 - 1 + m \text{ and } 0 \leq m < n_2 - n_1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The long exact sequences of Corollary 8.4 for $X = S^{n_1+m+q\alpha}$ with $0 \leq m < n_2 - n_1$ can be combined with the long exact sequence

$$\rightarrow 0 \rightarrow \mathbb{Z}_{(2)} \otimes \pi_{i+v\alpha}^{\mathbb{A}^1} S^{n_2+q\alpha} \rightarrow \mathbb{Z}_{(2)} \otimes \pi_{i+v\alpha}^{\mathbb{A}^1} S^{n_2+q\alpha} \rightarrow 0 \rightarrow \dots$$

associated to the $2\text{-}\mathbb{A}^1$ fiber sequence

$$S^{n_2+q\alpha} \rightarrow S^{n_2+q\alpha} \rightarrow *,$$

(which replaces in Theorem 8.5 the long exact sequences of Corollary 8.4 for $X = S^{n_1+m+q\alpha}$ with $n_2 - n_1 \leq m$) to form an exact couple. The E^1 -page equals to E^1 page of the EHP sequence constructed in Theorem 8.5 for $m < n_2 - n_1$, and $E_{i,m}^1 = 0$ for $m \geq n_2 - n_1$. The convergence is clear. \square

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